ON THE GAPS OF PRODUCTS OF MEMBERS OF A SEQUENCE WITH POSITIVE DENSITY

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1. Introduction

Throughout this paper we use the following notations: let $\mathbb{N}$ be the set of positive integers. The cardinality of a finite set S is denoted by $|S|$. If $\mathcal{A}$ is a subset of $\mathbb{N}$ then the counting function is $A(n) = \mathcal{A} \cap \{1, 2, \dots, n\}$. We call $\overline{d}(\mathcal{A}) = \limsup_{n\to\infty} \frac{A(n)}{n}$ and $\underline{d}(\mathcal{A}) = \liminf_{n\to\infty} \frac{A(n)}{n}$ the upper asymptotic density and lower asymptotic density of $\mathcal{A}$. For a given positive integer l and $\mathcal{A} \subset \mathbb{N}$, where $\mathcal{A} = a_1, a_2, \dots$ with $a_1 < a_2 < \dots$ denote by $\mathcal{A} + l$ the set of $\{a_1 + l, a_2 + l, \dots\}$ and by $\mathcal{A}_{\ge c}$ the set of $\{n : a_{n+1} - a_n \ge c\}$. The subsets $\mathcal{A}_{=c}$ and $\mathcal{A}_{<c}$ is defined similarly.

Let $\mathcal{A}$ be a given infinite subset of $\mathcal{N}$ with $\overline{d}(\mathcal{A}) = \alpha > 0$. Denote the set of products of the form $a_{i_1}a_{i_2}\dots a_{i_k}$ with $a_i \in \mathcal{A}$ by $B_{\mathcal{A}}^{(k)} = b_1^{(k)} < b_2^{(k)} < \dots$. A. Sárközy [2] formulated the following problem:

Is it true that for all $\alpha > 0$ there is a number $c = c(\alpha)$ depending only on α such that if $\mathcal{A} \subset \mathbb{N}$ is an infinite sequence whose lower asymptotic density $\underline{d}(\mathcal{A})$ is $> \alpha$, then $b_{n+1}^{(2)} - b_n^{(2)} \le c$ holds for nfintely many n? If the answer is affirmative, does this hold with $(\alpha) \le \frac{1}{\alpha^2}$ (If $k > 1$ is a natural number and $a_n = nk$, so $\alpha = \frac{1}{k}$, then $b_{n+1}^{(2)} - b_n^{(2)} = k^2$ for every n so this upper bound would be sharp.)

B. Bérczi [1] proved the existence of $c(\alpha)$ by showing that $c(\alpha) \le \frac{c_1}{\alpha^4}$ for some $c_1 > 0$. Improving this estimate we prove in this note that $c(\alpha) \le \frac{8}{\alpha^3}$.

Theorem 1.1. *Let $\mathcal{A}$ be a subset of positive integers with $\underline{d}(\mathcal{A}) = \alpha > 0$. Let us denote the sequence of the numbers of the form a_ia_j (with $a_i, a_j \in \mathcal{A}$) by $B_{\mathcal{A}}^{(2)} = b_1, b_2, \dots$, where $b_1 < b_2 \dots$. Then for infinitely many n we have $b_{n+1}^{(2)} - b_n^{(2)} < \frac{16}{\alpha^3}$.*

G. Brczi [1] asked the similar question about $B_{\mathcal{A}}^{(k)}$:

Let $\mathcal{A}$ be an infinite sequence of natural numbers with $\underline{d}(\mathcal{A}) = \alpha > 0$ and $k \ge 2$ a natural number. It is a natural question what can we say about the distribution of the products $a_{i_1}a_{i_2}\dots a_{i_k}$ where $a_{i_j} \in \mathcal{A}$. Let $B_{\mathcal{A}^{(k)}}$ these products as above. Is it true that there are infinitely many indices n with $b_{n+1}^{(k)} - b_n^{(k)} < c(k, \alpha)$ where this constant depends

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only on α, k? We have seen the case $k=2$ but unfortunately I cannot say anything about the other values of k, but the conjecture is surely true for every k.

In the next theorem we prove this conjecture for small ks.

Theorem 1.2. *For $k=2,3\dots,10,12$ and arbitrary $0<\alpha\le 1$ there exists a $c(k,\alpha)$ such that for every $\mathcal{A}\subset\mathbb{N}$ with $\underline{d}(\mathcal{A})=\alpha$ we have $b_{n+1}^{(k)}-b_n^{(k)}<c(k,\alpha)$ for infinitely many indices n.*

2. Proofs

The proof of Theorem 1.1 is based on the following lemma.

Lemma 2.1. *For every $0<\beta\le 1$ and subset $S\subset\mathbb{N}$ with $\overline{d}(\mathcal{S})=\beta$ there exists an integer $0<l<\frac{2}{\beta}$ such that $\overline{d}(\mathcal{S}\cap(\mathcal{S}+l)\ge\frac{\beta^2}{4}$.*

Proof. The definition of $\overline{d}(\mathcal{S})=\beta$ implies that there is an infinite sequence x_n ($x_n\to\infty$) such that $A(x_n)=(\beta+o(1))x_n$. It is not difficult to see that

$$(S_{\ge\frac{2}{\beta}}(x_n)-1)\frac{2}{\beta}\le x_n.$$

The right-hand side can be written as

$$S(x_n)-S_{<\frac{2}{\beta}}(x_n)-1)\frac{2}{\beta}=((\beta+o(1))x_n-S_{<\frac{2}{\beta}}(x_n))\frac{2}{\beta}=(2+o(1))x_n-S_{<\frac{2}{\beta}}(x_n),$$

therefore

$$(2+o(1))x_n-S_{<\frac{2}{\beta}}(x_n)\le x_n,$$

which implies

$$S_{<\frac{2}{\beta}}(x_n)\ge(\frac{\beta}{2}+o(1))x_n.$$

Hence $\overline{d}(S_{<\frac{2}{\beta}})\ge\frac{\beta}{2}$. Clearly $S_{<\frac{2}{\beta}}=\cup_{1\le<\frac{2}{\beta}}S_{=l}$. Hence the statement follows from the inequalities $\frac{\beta}{2}\le\overline{d}(S_{<\frac{2}{\beta}})\le\sum_{1\le l<\frac{2}{\beta}}\overline{d}(S_{=l})$. □

Proof of 2.1. By the previous lemma we have a $0<l<\frac{2}{\alpha}$ such that $\underline{d}(\mathcal{A}\cap(\mathcal{A}+l)\ge\frac{4}{\alpha^2}$ Picking up this l and denote by $\mathcal{C}=\mathcal{A}\cap(\mathcal{A}+l)$ we have $\overline{d}(C)\ge\alpha^2 4$. Using again the above lemma we get that there exist a $0<m<\frac{8}{\alpha^2}$ $\underline{d}(\mathcal{C}\cap(\mathcal{C}+m)\ge\frac{\alpha^4}{64}$. This means that for these l,m we have $\overline{d}(\mathcal{A}\cap(\mathcal{A}+l)\cap(\mathcal{A}+m)\cap(\mathcal{A}+m+l))\ge\frac{\alpha^4}{64}$. The proof follows from the inequalities $0<(a+l)(a+m)-(a+l+m)a=lm<\frac{16}{\alpha^3}$. □

Proof of Theorem 1.1. One of the classical diophantine problems is to find integers $e_1,e_2,\dots,e_k$ and $f_1,f_2,\dots,f_k$ satisfying

$$\sum_{i=1}^{k}e_i^s=\sum_{i=1}^{k}f_i^s\quad\text{for } s=1,2,\dots k-1 \tag{1}$$

(Prouhet-Tarry-Escott problem). We know that for $k = 2, 3, \ldots 10, 12$ there exist non-trivial solution of this problem, for instance:

$k = 2 : \{0, 2\}, \{1, 1\}$

$k = 3 : \{1, 2, 6\}, \{0, 4, 5\}$

$k = 4 : \{0, 4, 7, 11\}, \{1, 2, 9, 10\}$

$k = 5 : \{1, 2, 10, 14, 18\}, \{0, 4, 8, 16, 17\}$

$k = 6 : \{0, 4, 9, 17, 22, 26\}, \{1, 2, 12, 14, 24, 25\}$

$k = 7 : \{-51, -33 - -24, 7, 13, 38, 50\}, \{-50, -38, -13, -7, 24, 33, 51\}$

$k = 8 : \{0, 4, 9, 23, 27, 41, 46, 50\}, \{1, 2, 11, 20, 30, 39, 48, 49\}$

$k = 9 : \{1, 25, 31, 84, 87, 134, 158, 182, 198\}, \{2, 18, 42, 66, 113, 116, 169, 175, 199\}$

$k = 10 : \{-313, -301, -188, -100, -99, 99, 100, 188, 301, 313\},$

$\{-308, -307, -180, -131, -71, 71, 131, 180, 307, 308\}$

$k = 12 : \{0, 11, 24, 65, 90, 129, 173, 212, 237, 278, 291, 302\},$

$\{3, 5, 30, 57, 104, 116, 186, 198, 245, 272, 297, 299\}.$

One can easily to see that (1) is equivalent that

$$(x + e_1)(x + e_2) \ldots (x + e_k) - (x + f_1)(x + f_2) \ldots (x + f_k) = e_1 e_2 \ldots e_k - f_1 f_2 \ldots f_k = g_k.$$

Let $m_k = min\{e_1, e_2, \ldots, e_k, f_1, f_2, \ldots f_k\}$ and $M_k = max\{e_1, e_2, \ldots, e_k, f_1, f_2, \ldots f_k\}$. Szemerédi's theorem [3] implies that for every $0 < \alpha \le 1$ and positive integer N there exist positive constants $c_2(\alpha, N)$ such that for every subset $S \subset \mathbb{N}$ with $\overline{d}(S) = \alpha$ there exists a positive difference d such that $d \le c_2(\alpha, N)$ and there exists infinitely many $s \in \mathcal{S}$ such that $\{s, s+d, s+2d, \ldots, s+Nd\} \subset \mathcal{S}$. Hence we get that there exists infinitely many $a \in \mathcal{A}$ and $0 < d \le c_2(M_k - m_k, \alpha)$ such that $\{a+m_k d, a+(m_k+1)d, \ldots, a+(M_k)d\} \in \mathcal{A}$. Then $a + e_i d, a + f_i d \in \mathcal{A}$ for $1 \le i \le k$ and

$$(a + e_1 d)(a + e_2 d) \ldots (a + e_k d) - (a + f_1 d)(a + f_2 d) \ldots (a + f_k d) = g_k d^k.$$

Therefore $g_k c_2(M_k - m_k, \alpha)^k$ is suitable for $c(k, \alpha)$. □

References

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