An Extension of a Nathanson's Theorem on representation functions

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Abstract

For a given integer n and a set $\mathcal{S} \subseteq \mathbb{N}$ denote by $R^{(1)}_{h,\mathcal{S}}(n)$ the number of solutions of the equation $n = s_{i_1} + \cdots + s_{i_h}$, $s_{i_j} \in \mathcal{S}$, $j = 1, \ldots, h$. In this paper we determine all pairs $(\mathcal{A}, \mathcal{B})$, $\mathcal{A}, \mathcal{B} \subseteq \mathbb{N}$ for which $R^{(1)}_{h,\mathcal{A}}(n) = R^{(1)}_{h,\mathcal{B}}(n)$ from a certain point on, where h is a power of a prime. We also discuss the composite case.

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1 Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. For a given infinite set $\mathcal{A} \subset \mathbb{N}$ the representation functions $R^{(1)}_{h,\mathcal{A}}(n)$, $R^{(2)}_{h,\mathcal{A}}(n)$ and $R^{(3)}_{h,\mathcal{A}}(n)$ are defined in the following way:

$$R^{(1)}_{h,\mathcal{A}}(n) = \# \left\{ (a_{i_1}, \ldots, a_{i_h}) : a_{i_1} + \cdots + a_{i_h} = n, a_{i_1}, \ldots, a_{i_h} \in \mathcal{A} \right\},$$

$$R^{(2)}_{h,\mathcal{A}}(n) = \# \left\{ (a_{i_1}, \ldots, a_{i_h}) : a_{i_1} + \cdots + a_{i_h} = n, a_{i_1}, \ldots, a_{i_h} \in \mathcal{A}, a_{i_1} \leq \cdots \leq a_{i_h} \right\},$$

$$R^{(3)}_{h,\mathcal{A}}(n) = \# \left\{ (a_{i_1}, \ldots, a_{i_h}) : a_{i_1} + \cdots + a_{i_h} = n, a_{i_1}, \ldots, a_{i_h} \in \mathcal{A}, a_{i_1} < \cdots < a_{i_h} \right\}.$$

Representation functions have been extensively studied by many authors and are still a fruitful area of research in additive number theory. Using generating functions, Nathanson [7] proved the following result.

Let $F_{\mathcal{A}}$, $F_{\mathcal{B}}$ and T be finite sets of integers. If each residue class modulo m contains exactly the same number of elements of $F_{\mathcal{A}}$ as elements of $F_{\mathcal{B}}$, then we write $F_{\mathcal{A}} \equiv F_{\mathcal{B}} \pmod{m}$. If the number of solutions of the congruence $a + t \equiv n \pmod{m}$ with $a \in F_{\mathcal{A}}$, $t \in T$ equals to the number of solutions of the congruence $b + t \equiv n \pmod{m}$ with $b \in F_{\mathcal{B}}$, $t \in T$ for each residue class n modulo m then we write $F_{\mathcal{A}} + T \equiv F_{\mathcal{B}} + T \pmod{m}$.

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Nathanson's Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be infinite sets of nonnegative integers, $\mathcal{A} \neq \mathcal{B}$. Then $R^{(1)}_{2,\mathcal{A}}(n) = R^{(1)}_{2,\mathcal{B}}(n)$ from a certain point on if and only if there exist positive integers N, m and finite sets $F_{\mathcal{A}}$, $F_{\mathcal{B}}$, T with $F_{\mathcal{A}} \cup F_{\mathcal{B}} \subset \{0, 1, \dots, N\}$ and $T \subset \{0, 1, \dots, m-1\}$ such that $F_{\mathcal{A}} + T \equiv F_{\mathcal{B}} + T \pmod{m}$, and $\mathcal{A} = F_{\mathcal{A}} \cup \mathcal{C}$ and $\mathcal{B} = F_{\mathcal{B}} \cup \mathcal{C}$, where $\mathcal{C} = \{c > N | c \equiv t \pmod{m} \quad \text{for some} \quad t \in T\}$.*

It is clear that $R^{(2)}_{2,\mathcal{A}}(n) = \left\lceil \frac{R^{(1)}_{2,\mathcal{A}}(n)}{2} \right\rceil$ and $R^{(3)}_{2,\mathcal{A}}(n) = \left\lfloor \frac{R^{(1)}_{2,\mathcal{A}}(n)}{2} \right\rfloor$, thus for the sets $\mathcal{A}, \mathcal{B}$ in Nathanson's Theorem we have $R^{(2)}_{2,\mathcal{A}}(n) = R^{(2)}_{2,\mathcal{B}}(n)$ and $R^{(3)}_{2,\mathcal{A}}(n) = R^{(3)}_{2,\mathcal{B}}(n)$ from a certain point on. It is easy to see that the symmetric difference of the sets $\mathcal{A}$ and $\mathcal{B}$ in the above theorem is finite. A. Sárközy asked whether there exist two infinite sets of nonnegative integers $\mathcal{A}$ and $\mathcal{B}$ with infinite symmetric difference, i.e.

$$|(\mathcal{A} \cup \mathcal{B}) \setminus (\mathcal{A} \cap \mathcal{B})| = \infty$$

and

$$R^{(i)}_{2,\mathcal{A}}(n) = R^{(i)}_{2,\mathcal{B}}(n)$$

if $n \geq n_0$, for i = 1, 2, 3. For $i = 1$ the answer is negative (see in [3]). For $i = 2$ G. Dombi [3] and for $i = 3$ Y. G. Chen and B. Wang [2] proved that the set of nonnegative integers can be partitioned into two subsets $\mathcal{A}$ and $\mathcal{B}$ such that $R^{(i)}_{2,\mathcal{A}}(n) = R^{(i)}_{2,\mathcal{B}}(n)$ for all $n \geq n_0$. In [6] Lev gave a common proof to the above mentioned results of Dombi [3] and Chen and Wang [2]. Using generating functions Cs. Sándor [8] determined the sets $\mathcal{A} \subset \mathbb{N}$ for which either

$$R^{(2)}_{2,\mathcal{A}}(n) = R^{(2)}_{2,\mathbb{N}\setminus\mathcal{A}}(n) \quad \text{for all} \quad n \geq n_0$$

or

$$R^{(3)}_{2,\mathcal{A}}(n) = R^{(3)}_{2,\mathbb{N}\setminus\mathcal{A}}(n) \quad \text{for all} \quad n \geq n_0.$$

In [9] M. Tang gave an elementary proof of Cs. Sándor's results. Y. G. Chen and M. Tang studied related questions in [1] .

Let $\mathcal{C}$ be a finite set of integers. Let $F_{\mathcal{A}}(z)$, $F_{\mathcal{B}}$, $T(z)$ denote polynomials and $A(z)$, $B(z)$ denote power series having coefficients from the set $\mathcal{C}$ (i.e. $A(z) = \sum_{n=0}^{\infty} a_n z^n$, where $a_n \in \mathcal{C}$ and z is a complex number, $z = r \cdot e^{2\pi i\theta}$). These series converge in the open unit disc. If $\mathcal{C} = \{0, 1\}$ then the generating functions of the sets $\mathcal{A}$, $\mathcal{B}$, $F_{\mathcal{A}}$, $F_{\mathcal{B}}$ and $T \subseteq \mathbb{N}$ are special kind of these polynomials and power series.

To make the proofs easier we will use the following notation: $A(z) \sim B(z)$ means that $A(z) - B(z)$ is a polynomial. Using these we can rewrite Nathanson's Theorem in equivalent form:

Equivalent form of Nathanson's Theorem. *Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$, $B(z) = \sum_{n=0}^{\infty} b_n z^n$, $a_n, b_n \in \{0, 1\}$. Then $A(z)^2 \sim B(z)^2$ if and only if there exist positive integers N_0, m and polynomials $F_{\mathcal{A}}(z) = \sum_{n=0}^{mN_0-1} d_n z^n$, $F_{\mathcal{B}}(z) = \sum_{n=0}^{mN_0-1} e_n z^n$, $d_n, e_n \in \{0, 1\}$ and $T(z) = \sum_{n=0}^{m-1} t_n z^n$,*

$t_n \in \{0,1\}$ *such that*

$$A(z) = F_{\mathcal{A}}(z) + \frac{T(z)z^{mN_0}}{1-z^m},$$
$$B(z) = F_{\mathcal{B}}(z) + \frac{T(z)z^{mN_0}}{1-z^m}$$

and

$$1-z^m | T(z)\left(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)\right)$$

holds.

S. Z. Kiss, R. Rozgonyi and Cs. Sándor [4] conjectured that Nathanson's theorem can be generalized as follows.

Conjecture. *Let $h \geq 2$, $\mathcal{A}$ and $\mathcal{B}$ be infinite sets of nonnegative integers, $\mathcal{A} \neq \mathcal{B}$. Then $R^{(1)}_{h,\mathcal{A}}(n) = R^{(1)}_{h,\mathcal{B}}(n)$ from a certain point on if and only if there exist positive integers N_0, m and sets $F_{\mathcal{A}}$, $F_{\mathcal{B}}$ and T such that $F_{\mathcal{A}} \cup F_{\mathcal{B}} \subset \{0,1,\dots,mN_0-1\}$, $T \subset \{0,1.\dots,m-1\}$,*

$$\mathcal{A} = F_{\mathcal{A}} \cup \{km + t : k \geq N_0, t \in T\},$$
$$\mathcal{B} = F_{\mathcal{B}} \cup \{km + t : k \geq N_0, t \in T\},$$

and

$$(1-z^m)^{h-1} \left| T(z)^{h-1}\left(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)\right)\right.,$$

where $F_{\mathcal{A}}(z)$, $F_{\mathcal{B}}(z)$ and $T(z)$ denote the generating functions of the sets $F_{\mathcal{A}}$, $F_{\mathcal{B}}$ and T.

In [4] they proved the sufficiency part of the Conjecture, and they also proved the Conjecture for the case $h = 3$.
Using power series, we can rewrite the Conjecture in equivalent form.

Equivalent form of the Conjecture. *Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$, $B(z) = \sum_{n=0}^{\infty} b_n z^n$, $a_n, b_n \in \{0,1\}$. Then $A(z)^h \sim B(z)^h$ if and only if there exist positive integers N_0, m and polynomials $F_{\mathcal{A}}(z) = \sum_{n=0}^{mN_0-1} d_n z^n$, $F_{\mathcal{B}}(z) = \sum_{n=0}^{mN_0-1} e_n z^n$, $d_n, e_n \in \{0,1\}$ and $T(z) = \sum_{n=0}^{m-1} t_n z^n$, $t_n \in \{0,1\}$ such that*

$$A(z) = F_{\mathcal{A}}(z) + \frac{T(z)z^{mN_0}}{1-z^m},$$
$$B(z) = F_{\mathcal{B}}(z) + \frac{T(z)z^{mN_0}}{1-z^m}$$

and

$$(1-z^m)^{h-1} \left| T(z)^{h-1}\left(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)\right)\right.$$

holds.

In this paper we prove the above Conjecture in the case $h = p^s$, where p is prime.

Theorem 1. *Let $h = p^s$ and let $\mathcal{C} \subseteq \mathbb{Z}$ be a finite set which contains incongruent integers modulo p. Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$, $B(z) = \sum_{n=0}^{\infty} b_n z^n$ be power series, where $a_n, b_n \in \mathcal{C}$. Then $A(z)^h \sim B(z)^h$ if and only if there exist positive integers N_0, m and polynomials $F_{\mathcal{A}}(z) = \sum_{n=0}^{mN_0-1} d_n z^n$, $F_{\mathcal{B}}(z) = \sum_{n=0}^{mN_0-1} e_n z^n$, $d_n, e_n \in \mathcal{C}$ and $T(z) = \sum_{n=0}^{m-1} t_n z^n$, $t_n \in \mathcal{C}$ such that*

$$A(z) = F_{\mathcal{A}}(z) + \frac{T(z) z^{mN_0}}{1 - z^m},$$
$$B(z) = F_{\mathcal{B}}(z) + \frac{T(z) z^{mN_0}}{1 - z^m}$$

and

$$(1 - z^m)^{h-1} | T(z)^{h-1} \left(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)\right)$$

holds.

Corollary. *The case $\mathcal{C} = \{0, 1\}$ implies that the Conjecture is true for the case $h = p^s$.*

In order to prove Theorem 1. we verify the following three lemmas.

Lemma 1. *Let $\mathcal{C}$ be a set of integers. Suppose that there exist positive integers N_0, m and polynomials $F_{\mathcal{A}}(z) = \sum_{n=0}^{mN_0-1} d_n z^n$, $F_{\mathcal{B}}(z) = \sum_{n=0}^{mN_0-1} e_n z^n$, $d_n, e_n \in \mathcal{C}$ and $T(z) = \sum_{n=0}^{m-1} t_n z^n$, $t_n \in \mathcal{C}$ such that*

$$A(z) = F_{\mathcal{A}}(z) + \frac{T(z) z^{mN_0}}{1 - z^m},$$
$$B(z) = F_{\mathcal{B}}(z) + \frac{T(z) z^{mN_0}}{1 - z^m}$$

and

$$(1 - z^m)^{h-1} \left| T(z)^{h-1} \left(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)\right)\right.$$

holds. Then $A(z)^h \sim B(z)^h$.

The following example shows that for any $\mathcal{C} \subseteq \mathbb{Z}$, $|\mathcal{C}| \geq 2$ and $h \geq 2$ there exist different power series $A(z), B(z)$ having their coefficients from $\mathcal{C}$ with the property $A(z)^h \sim B(z)^h$.

Proposition 1. *Let $\mathcal{C} \subseteq \mathbb{Z}$, $|\mathcal{C}| \geq 2$. Then there exist series $A(z) = \sum_{n=0}^{\infty} a_n z^n$, $B(z) = \sum_{n=0}^{\infty} b_n z^n$, $a_n, b_n \in \mathcal{C}$ such that $A(z) \neq B(z)$ and $A(z)^h \sim B(z)^h$.*

In the proof of Theorem 1. we only use the fact that h is a power of prime in Lemma 2.

Lemma 2. *Let $h = p^s$ and let $\mathcal{C} \subseteq \mathbb{Z}$, where no element of $\mathcal{C}$ are congruent modulo p. Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$, $B(z) = \sum_{n=0}^{\infty} b_n z^n$, $a_n, b_n \in \mathcal{C}$. The condition $A(z)^h \sim B(z)^h$ implies that $A(z) \sim B(z)$.*

The condition of Lemma 2. that $\mathcal{C}$ contains incongruent integers modulo p is important, because Imre Z. Ruzsa gave the identity

$$\left(-1 + \frac{2z^4}{1-z^2}\right)^2 - \left(\frac{2z^5}{1-z^2}\right)^2 = 1 - 4z^4 + 4z^6.$$

It means that there exist power series $A(z)$ and $B(z)$ having coefficients from the set $\mathcal{C} = \{-1, 0, 2\}$ such that $A(z)^2 \sim B(z)^2$, but $A(z) \not\sim B(z)$, because

$$-1 + \frac{2z^4}{1-z^2} - \frac{2z^5}{1-z^2} = -1 + 2\sum_{n=4}^{\infty}(-1)^n z^n.$$

We can generalize Imre Z. Ruzsa's construction in the following way:

Proposition 2. *Let h be a prime number. Then there exist a set $\mathcal{C}_h = \{c_1, c_2, \dots, c_{h+1}\}$, $c_1, \dots, c_{h+1} \in \mathbb{Z}$ such that $c_1, \dots c_h$ form a complete set of residues modulo h and power series $A(z) = \sum_{n=0}^{\infty} a_n z^n$, $B(z) = \sum_{n=0}^{\infty} b_n z^n$, $a_n, b_n \in \mathcal{C}$ such that $A(z)^h \sim B(z)^h$ but $A(z) \not\sim B(z)$.*

Problem. *Let $\mathcal{C} \subset \mathbb{Z}$ be a finite set, $A(z) = \sum_{n=0}^{\infty} a_n z^n$, $B(z) = \sum_{n=0}^{\infty} b_n z^n$, $a_n, b_n \in \mathcal{C}$. Is it true that the condition $A(z)^h \sim B(z)^h$ implies that the coefficients of the power series $A(z)$ and $B(z)$ are periodic?*

Lemma 3. *Let $\mathcal{C}$ be a finite set of integers, $A(z) = \sum_{n=0}^{\infty} a_n z^n$, $B(z) = \sum_{n=0}^{\infty} b_n z^n$, $a_n, b_n \in \mathcal{C}$. If $A(z)^h \sim B(z)^h$ and $A(z) \sim B(z)$ holds, then there exist positive integers N_0, m and polynomials $F_{\mathcal{A}}(z) = \sum_{n=0}^{mN_0-1} d_n z^n$, $F_{\mathcal{B}}(z) = \sum_{n=0}^{mN_0-1} e_n z^n$, $d_n, e_n \in \mathcal{C}$ and $T(z) = \sum_{n=0}^{m-1} t_n z^n$, $t_n \in \mathcal{C}$ such that*

$$A(z) = F_{\mathcal{A}}(z) + \frac{T(z)z^{mN_0}}{1-z^m},$$
$$B(z) = F_{\mathcal{B}}(z) + \frac{T(z)z^{mN_0}}{1-z^m}$$

and

$$(1-z^m)^{h-1} \left| T(z)^{h-1} \left(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)\right)\right. . \tag{1}$$

2 Proofs

Proof of Lemma 1. We have to show that $A(z)^h \sim B(z)^h$, that is $A(z)^h - B(z)^h = P(z)$, where $P(z)$ is a polynomial. Using the binomial theorem and the assumptions of Lemma 1 we get that

$$A(z)^h - B(z)^h = \left(F_{\mathcal{A}}(z) + \frac{T(z)z^{mN_0}}{1-z^m}\right)^h - \left(F_{\mathcal{B}}(z) + \frac{T(z)z^{mN_0}}{1-z^m}\right)^h =$$

$$= \sum_{i=1}^{h} \binom{h}{i} \left(F_{\mathcal{A}}(z)^i - F_{\mathcal{B}}(z)^i\right) \frac{T(z)^{h-i} \cdot z^{(h-i)mN_0}}{(1-z^m)^{h-i}}.$$

Since

$$F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z) \,\big|\, F_{\mathcal{A}}(z)^i - F_{\mathcal{B}}(z)^i \,,$$

so

$$T(z)^{h-i} \left(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)\right) \big| \left(F_{\mathcal{A}}(z)^i - F_{\mathcal{B}}(z)^i\right) T(z)^{h-i} \cdot z^{(h-i)mN_0}.$$

Therefore it is enough to show that for every $1 \le i \le h-1$ we have

$$(1-z^m)^{h-i} \,\big|\, T(z)^{h-i} \left(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)\right). \tag{2}$$

From the assumptions we know that $(1-z^m)^{h-1} \,\big|\, T^{h-1}(z) \left(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)\right)$ holds.

For a given integer d let denote by $\phi_d(z)$ the dth cyclotomic polynomial. It remains to prove that

$$\phi_d(z)^{h-i} \,\big|\, T(z)^{h-i} \left(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)\right). \tag{3}$$

Let $T(z) = \phi_d(z)^{k_1} u(z)$ and $F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z) = \phi_d(z)^{k_2} v(z)$, where $u(z)$ and $v(z)$ are polynomials with the property $\phi_d(z) \nmid u(z)v(z)$. By assumptions of Lemma 1 we know that $(h-1)k_1 + k_2 \ge h-1$. Thus either $k_1 = 0$, then $k_2 \ge h-1$, therefore

$$\phi_d(z)^{h-i} \mid F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)$$

or $k_1 \ge 1$, therefore $\phi_d(z) \mid T(z)$ so

$$\phi_d(z)^{h-i} \mid T(z)^{h-i}.$$

This completes the proof of Lemma 1. ■

Proof of Proposition 1: Let $\text{Bin}(i)$ denote the parity of the number of 1's in the binary form of i. (i.e. $\text{Bin}(i) = 1$, if the number of 1s in the binary form of i is even and $\text{Bin}(i) = -1$, if the number of the 1-s in the binary form of i is odd.)

It is easy to see that $\displaystyle\prod_{i=0}^{h-2}(1-z^{2^i}) = \sum_{i=0}^{2^{h-1}-1} \text{Bin}(i) z^i$.

Suppose that $c_1, c_2 \in \mathcal{C}$. Let $A(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$, where for $n \le 2^{h-1}-1$

$$a_n = \begin{cases} c_1 & \text{if Bin(n)} = 1 \\ c_2 & \text{if Bin(n)} = -1, \end{cases}$$

and $a_n = c_1$, for $n \geq 2^{h-1}$.
Let $B(z) = \sum_{n=0}^{\infty} b_n z^n$, where for $n \leq 2^{h-1} - 1$

$$b_n = \begin{cases} c_2 & \text{if Bin(n)} = 1 \\ c_1 & \text{if Bin(n)} = -1, \end{cases}$$

and $b_n = c_1$, for $n \geq 2^{h-1}$.

Then

$$A(z) = F_{\mathcal{A}}(z) + \frac{c_1 z^{2^{h-1}}}{1-z},$$

and

$$B(z) = F_{\mathcal{B}}(z) + \frac{c_1 z^{2^{h-1}}}{1-z}.$$

We may apply Lemma 1 with $m = 1$ and $F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z) = (c_1 - c_2)\prod_{i=0}^{h-2}(1 - z^{2^i})$ and $T(z) = c_1$, because

$$(1 - z^m)^{h-1} \,\big|\, T(z)^{h-1}(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z))$$

is equivalent with

$$(1-z)^{h-1} \Big| c_1^{h-1}(c_1 - c_2)\prod_{i=0}^{h-2}(1 - z^{2^i}),$$

which is obviously true. ■

Proof of Lemma 2. It is clear that

$$A(z)^h = \left(\sum_{n=0}^{\infty} a_n z^n\right)^h = \sum_{n=0}^{\infty} g_n z^n,$$

where

$$g_n = \sum_{\substack{(i_1, i_2, \dots, i_h) \\ i_1 + i_2 + \dots + i_h = n}} a_{i_1} a_{i_2} \dots a_{i_h} = \sum_{t=1}^{h} \sum_{\substack{(j_1, \dots, j_t) \\ j_i \in \mathbb{N} \\ 0 \leq j_1 < j_2 < \dots < j_t}} \sum_{\substack{(m_1, \dots, m_t), m_i \in \mathbb{Z}^+ \\ m_1 + \dots + m_t = h \\ m_1 j_1 + \dots + m_t j_t = n}} \frac{h!}{m_1!\, m_2! \dots m_t!} a_{j_1}^{m_1} a_{j_2}^{m_2} \dots a_{j_t}^{m_t}. \tag{4}$$

Using (4), the formula of Legendre and Fermat's Theorem we get that modulo p we have

$$\frac{h!}{m_1!\, m_2! \dots m_t!} a_{j_1}^{m_1} a_{j_2}^{m_2} \dots a_{j_t}^{m_t} = \frac{p^s!}{m_1!\, m_2! \dots m_t!} a_{j_1}^{m_1} a_{j_2}^{m_2} \dots a_{j_t}^{m_t} \equiv$$

$$\equiv \begin{cases} 0 & \text{for} \quad t \geq 2 \\ a_{i_1}^{p^s} = a_{\frac{n}{p^s}}^{p^s} \equiv a_{\frac{n}{p^s}} & \text{for} \quad t = 1. \end{cases} \tag{5}$$

So $g_n \equiv 0 \pmod{p}$ if $p^s \not| n$ and $g_n \equiv a_{\frac{n}{p^s}} \pmod{p}$, if $p^s \mid n$.
Similarly for $B(z)^h$ we get that

$$B(z)^h = \left(\sum_{n=0}^{\infty} b_n z^n\right)^h = \sum h_n z^n,$$

where $h_n \equiv 0 \pmod{p}$ if $p^s \not| n$ and $h_n \equiv b_{\frac{n}{p^s}} \pmod{p}$, if $p^s | n$.
We know that $g_n \equiv h_n \pmod{p}$ for $n \geq n_0$. For $np^s \geq n_0$ the congruence $g_{np^s} \equiv h_{np^s} \pmod{p}$ implies that $a_n \equiv b_n \pmod{p}$. Using the condition that the set $\mathcal{C}$ contains incongruent integers modulo p we get that $a_n = b_n$ for $n \geq n_0/p^s$. This completes the proof of Lemma 2. ■

Proof of Lemma 3. Since $A(z) \sim B(z)$ and $A(z)^h \sim B(z)^h$ we can write that $A(z) = B(z) + P(z)$ and $A(z)^h = B(z)^h + Q(z)$, where $P(z)$ and $Q(z)$ are polynomials. Thus

$$Q(z) = A(z)^h - B(z)^h = (B(z) + P(z))^h - B(z)^h = \sum_{i=1}^{h} \binom{h}{i} P(z)^i B(z)^{h-i}.$$

Multiply both side by $P(z)^{h-2}$, we get that

$$Q(z)P(z)^{h-2} = (P(z)B(z))^{h-1} \left(h + \frac{\binom{h}{2}P(z)}{B(z)} + \frac{\binom{h}{3}P(z)^2}{B(z)^2} + \dots \right). \tag{6}$$

We show that $P(z)B(z)$ is bounded in the open unit disc ($|z| < 1, z \in \mathbb{C}$). Assume, that it is not true. Then there is an infinite sequence $z_1, z_2, \dots, z_n \dots$, $|z_n| < 1$ for which $|P(z_n)B(z_n)| \to \infty$ (so $|B(z_n)| \to \infty$), and thus

$$\left|Q(z_n)P(z_n)^{h-2}\right| = \left|(P(z_n)B(z_n))^{h-1} \left(h + \frac{\binom{h}{2}P(z_n)}{B(z_n)} + \frac{\binom{h}{3}P(z_n)^2}{B(z_n)^2} + \dots \right)\right| \to \infty,$$

while the left hand side is bounded. So $P(z)B(z)$ is bounded in the open unit disc.

Next we show, that $P(z)B(z)$ is a polynomial. Suppose indirectly, that $P(z)B(z) = \sum_{k=0}^{\infty} k_n z^n$, $k_n \in \mathbb{Z}$ and $k_n \neq 0$ for infinitely many integers n. Let $z = re^{2\pi i\varphi}$. For $r < 1$ the Parseval-formula gives

$$\int_0^1 \left|P(re^{2\pi i\varphi})B(re^{2\pi i\varphi})\right|^2 \mathrm{d}\varphi = \sum_{n=0}^{\infty} k_n^2 r^{2n}. \tag{7}$$

Since $P(z)B(z)$ is bounded, if $r \to 1^-$ then the left-hand side of (7) is bounded, but the right-hand side of (7) tends to infinity. So $P(z)B(z) = R(z)$, where $R(z)$ is a polynomial.
Write $P(z)$ in the form $P(z) = \sum_{i=0}^{m} \lambda_i z^i$. Then b_n fulfils the following linear recursion when $n \geq n_1$

$$b_n = -\frac{1}{\lambda_0}\left(\lambda_1 b_{n-1} + \lambda_2 b_{n-2} + \dots + \lambda_m b_{n-m}\right).$$

This means, that b_n is determinded by $b_{n-1}, b_{n-2}, \ldots, b_{n-m}$. Since we can choose b_n in finitely many way we conclude that b_n is periodic. Therefore there exist positive integers N_0, m and series $F_{\mathcal{A}}(z) = \sum_{n=0}^{mN_0-1} d_n z^n$, $F_{\mathcal{B}}(z) = \sum_{n=0}^{mN_0-1} e_n z^n$, $d_n, e_n \in \mathcal{C}$ and $T(z) = \sum_{n=0}^{m-1} t_n z^n$, $t_n \in \mathcal{C}$ such that

$$A(z) = F_{\mathcal{A}}(z) + \frac{T(z)z^{mN_0}}{1-z^m},$$
$$B(z) = F_{\mathcal{B}}(z) + \frac{T(z)z^{mN_0}}{1-z^m}.$$

In the last step we have to show (1). We show by induction on k that for every $1 \le k \le h-1$

$$(1-z^m)^k \left| T(z)^k \left(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)\right) \right. \tag{8}$$

holds. It is clear that

$$A(z)^h - B(z)^h = \left(F_{\mathcal{A}}(z) + \frac{T(z)z^{mN_0}}{1-z^m}\right)^h - \left(F_{\mathcal{B}}(z) + \frac{T(z)z^{mN_0}}{1-z^m}\right)^h = Q(z),$$

where $Q(z)$ is a polynomial. Using the binomial theorem we get

$$(1-z^m)^h \left| \left(F_{\mathcal{A}}(z)(1-z^m) + T(z)z^{mN_0}\right)^h - \left(F_{\mathcal{B}}(z)(1-z^m) + T(z)z^{mN_0}\right)^h = \right.$$
$$= \sum_{i=0}^{h} \binom{h}{i} T^i(z) z^{imN_0} \left(F_{\mathcal{A}}(z)^{h-i} - F_{\mathcal{B}}(z)^{h-i}\right)(1-z^m)^{h-i}. \tag{9}$$

Obviously if in (9) $i=0$ then $(1-z^m)^h \left| \binom{h}{0} T(z)^0 z^{0\cdot mN_0} \left(F_{\mathcal{A}}(z)^h - F_{\mathcal{B}}(z)^h\right)(1-z^m)^h \right.$ and also if in (9) $i=h$ then $(1-z^m)^h \left| \binom{h}{h} T^h(z) z^{hmN_0} \cdot \left(F_{\mathcal{A}}(z)^0 - F_{\mathcal{B}}(z)^0\right)(1-z^m)^0 \right.$ holds. So (9) is equivalent to

$$(1-z^m)^h \left| \sum_{i=1}^{h-1} \binom{h}{i} T(z)^i z^{imN_0} \left(F_{\mathcal{A}}(z)^{h-i} - F_{\mathcal{B}}^{h-i}(z)\right)(1-z^m)^{h-i}. \right. \tag{10}$$

At first let $k=1$. In (10) if $i \le h-2$ then

$$(1-z^m)^2 \left| \binom{h}{i} z^{imN_0} T(z)^i \left(F_{\mathcal{A}}(z)^{h-i} - F_{\mathcal{B}}(z)^{h-i}\right)(1-z^m)^{h-i}. \right. \tag{11}$$

So in (10) if $i = h-1$ then

$$(1-z^m)^2 \left| \binom{h}{h-1} z^{(h-1)mN_0} T(z)^{h-1} \left(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)\right)(1-z^m) \right. \tag{12}$$

should be also true. Since $(z, 1-z^m) = 1$ from (12) we have

$$1-z^m \left| T(z)^{h-1} \left(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)\right). \right. \tag{13}$$

We know, that $1 - z^m = -\prod_{d|m} \phi_d(z)$, where $\phi_d(z)$ denotes the dth cyclotomic polynomial. $1 - z^m$ has no multiple root, which implies

$$1 - z^m \left| T(z) \left(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z) \right) . \right. \tag{14}$$

Now assume that for $1 \le l \le k-1$, $k \le h-1$ we have

$$(1 - z^m)^l \left| T(z)^l \left(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z) \right) . \right.$$

We have to prove that

$$(1 - z^m)^k \left| T(z)^k \left(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z) \right) . \right. \tag{15}$$

In (10)

$$T(z)^i \left(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z) \right) (1 - z^m)^{h-i} \left| \binom{h}{i} z^{imN_0} T(z)^i \left(F_{\mathcal{A}}(z)^{h-i} - F_{\mathcal{B}}(z)^{h-i} \right) (1 - z^m)^{h-i} . \right. \tag{16}$$

Using the assumption of the induction we get

$$(1 - z^m)^{h-i+\min\{k-1,i\}} \left| T(z)^i \left(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z) \right) (1 - z^m)^{h-i} . \right. \tag{17}$$

From (16) and (17) we obtain that

$$(1 - z^m)^{h-i+\min\{k-1,i\}} \left| \binom{h}{i} z^{imN_0} T(z)^i \left(F_{\mathcal{A}}(z)^{h-i} - F_{\mathcal{B}}(z)^{h-i} \right) (1 - z^m)^{h-i} . \right. \tag{18}$$

So from (10) we know

$$(1 - z^m)^{k+1} \left| (1 - z^m)^h \right| \sum_{i=1}^{h-1} \binom{h}{i} T(z)^i z^{imN_0} \left(F_{\mathcal{A}}(z)^{h-i} - F_{\mathcal{B}}(z)^{h-i} \right) (1 - z^m)^{h-i} . \tag{19}$$

If $\min\{k-1, i\} = i$, which means $i \le k-1$ then $h - i + \min\{k-1, i\} = h \ge k+1$. Thus

$$(1 - z^m)^{k+1} \mid \binom{h}{i} T(z)^i z^{imN_0} \left(F_{\mathcal{A}}(z)^{h-i} - F_{\mathcal{B}}(z)^{h-i} \right) (1 - z^m)^{h-i} \tag{20}$$

holds.
If $\min\{k-1, i\} = k-1$, then $k-1 \le i \le h-1$. If $k-1 \le i \le h-2$, then $h - i + \min\{k-1, i\} \ge 2 + k - 1 = k+1$. Thus

$$(1 - z^m)^{k+1} \mid \binom{h}{i} T(z)^i z^{imN_0} \left(F_{\mathcal{A}}(z)^{h-i} - F_{\mathcal{B}}(z)^{h-i} \right) (1 - z^m)^{h-i} . \tag{21}$$

So (10), (20) and (21) imply that for $i = h-1$ we have

$$(1 - z^m)^{k+1} \left| \binom{h}{h-1} T(z)^{h-1} z^{(h-1)mN_0} \left(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z) \right) (1 - z^m) , \right. \tag{22}$$

which means that

$$(1 - z^m)^k \left| T(z)^{h-1} \left(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z) \right) . \right. \tag{23}$$

We use, that $1 - z^m = -\prod_{d|m} \phi_d(z)$, where $\phi_d(z)$ denotes the dth cyclotomic polynomial.
This means that for every d, $d\,|m$ we have $\phi_d(z)^k \mid T(z)^{h-1}\left(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)\right)$.
If $\phi_d(z) \mid T(z)$ then $\phi_d(z)^k \mid T(z)^k$ is also true. If $\phi_d(z) \not| T(z)$ then $\phi_d(z)^k \mid F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)$.
So for every $d \mid m$, $\phi_d(z)^k \mid T(z)^k \left(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)\right)$ which means that

$$(1 - z^m)^k \mid T(z)^k \left(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)\right). \tag{24}$$

This ends the the induction and the proof of Lemma 3.■

Proof of Proposition 2. Let

$$a'_i = \frac{\prod_{j=0}^{i-1}(h^2 + 1 - jh)}{i!} \cdot (-1)^i h^{h-1-i} - (-1)^i \binom{h}{i} h^{h-1},$$

for $1 \le i \le h - 1$ and

$$A(z) = \sum_{i=1}^{h-1} a'_i (1 - z^h)^{i-1} + \frac{h^{h-1} z^{h^2}}{1 - z^h},$$

$$B(z) = \frac{h^{h-1} z^{h^2+1}}{1 - z^h}.$$

We will show that

1. $A(z)^h \sim B(z)^h$
2. There exists a set of integers $\mathcal{C}_h = \{c_1, c_2, \dots, c_{h+1}\}$ such that $c_1, \dots c_h$ form a complete set of residues modulo h and for the previous power series $A(z) = \sum_{n=0}^{\infty} a_n z^n$, $B(z) = \sum_{n=0}^{\infty} b_n z^n$ we have $a_n, b_n \in \mathcal{C}_h$.

It is easy to see that $A(z) \not\sim B(z)$.
First we prove statement 1.

$$A(z)^h - B(z)^h = \left(\sum_{i=1}^{h-1} a'_i (1 - z^h)^{i-1} + \frac{h^{h-1} z^{h^2}}{1 - z^h} \right)^h - \left(\frac{h^{h-1} z^{h^2+1}}{1 - z^h} \right)^h. \tag{25}$$

From (25) we see that to prove $A(z)^h \sim B(z)^h$ we need to show

$$(1 - z^h)^h \Bigg| \left(\sum_{i=1}^{h-1} a'_i (1 - z^h)^i + h^{h-1} z^{h^2} \right)^h - \left(h^{h-1} z^{h^2+1} \right)^h. \tag{26}$$

Let $x = 1 - z^h$. (26) is equivalent to the following

$$x^h | \left(\sum_{i=1}^{h-1} a'_i x^i + h^{h-1} (1 - x)^h \right)^h - h^{h^2-h} (1 - x)^{h^2+1}. \tag{27}$$

So it is enough to show (27). For $-1 < x < 1$ by the binomial series expansion we get

$$h^{h^2-h}(1-x)^{h^2+1} = \left(h^{h-1}(1-x)^{\frac{h^2+1}{h}}\right)^h = \tag{28}$$

$$\left(h^{h-1}\sum_{i=0}^{\infty}\binom{\frac{h^2+1}{h}}{i}(-1)^i x^i\right)^h = \left(h^{h-1}\sum_{i=0}^{h-1}\binom{\frac{h^2+1}{h}}{i}(-1)^i x^i + h^{h-1}\sum_{i=h}^{\infty}\binom{\frac{h^2+1}{h}}{i}(-1)^i x^i\right)^h =$$

$$= \left(\sum_{i=0}^{h-1} h^{h-1-i}\frac{\prod_{j=0}^{i-1}(h^2+1-jh)}{i!}(-1)^i x^i + h^{h-1}\sum_{i=h}^{\infty}\binom{\frac{h^2+1}{h}}{i}(-1)^i x^i\right)^h =$$

$$\left(\sum_{i=0}^{h-1} h^{h-1-i}\frac{\prod_{j=0}^{i-1}(h^2+1-jh)}{i!}(-1)^i x^i\right)^h +$$

$$+\sum_{k=1}^{h}\binom{h}{k}\left(\sum_{i=0}^{h-1} h^{h-1-i}\frac{\prod_{j=0}^{i-1}(h^2+1-jh)}{i!}(-1)^i x^i\right)^{h-k}\left(h^{h-1}\sum_{i=h}^{\infty}\binom{\frac{h^2+1}{h}}{i}(-1)^i x^i\right)^k =$$

$$= \left(\sum_{i=0}^{h-1} h^{h-1-i}\frac{\prod_{j=0}^{i-1}(h^2+1-jh)}{i!}(-1)^i x^i\right)^h + \sum_{i=h}^{\infty} d_i x^i.$$

Thus

$$h^{h^2-h}(1-x)^{h^2+1} - \left(\sum_{i=0}^{h-1} h^{h-1-i}\frac{\prod_{j=0}^{i-1}(h^2+1-jh)}{i!}(-1)^i x^i\right)^h = \sum_{i=h}^{\infty} d_i x^i = \sum_{i=h}^{h^2+1} d_i x^i. \tag{29}$$

Hence

$$x^h \mid \left(\sum_{i=0}^{h-1} h^{h-1-i}\frac{\prod_{j=0}^{i-1}(h^2+1-jh)}{i!}(-1)^i x^i\right)^h - h^{h^2-h}(1-x)^{h^2+1}. \tag{30}$$

Since for every $1 \le i \le h-1$

$$(-1)^i h^{h-1-i}\frac{\prod_{j=0}^{i-1}(h^2+1-jh)}{i!} = a'_i + (-1)^i h^{h-1}\binom{h}{i},$$

from (30) we get that

$$x^h \mid \left(\sum_{i=1}^{h-1} a'_i x^i + h^{h-1} + h^{h-1}\sum_{i=1}^{h-1}(-1)^i\binom{h}{i}x^i\right)^h - h^{h^2-h}(1-x)^{h^2+1}. \tag{31}$$

Thus

$$x^h \,\Big|\, \left(\sum_{i=1}^{h-1} a'_i x^i + h^{h-1}\sum_{i=0}^{h}(-1)^i\binom{h}{i}x^i\right)^h - h^{h^2-h}(1-x)^{h^2+1} \tag{32}$$

is also true. Using the binomial theorem we get that $\sum_{i=0}^{h}(-1)^i\binom{h}{i}x^i = (1-x)^h$, which proves statement 1.

Now we prove statement 2. We will show that $\frac{\prod_{j=0}^{i-1}(h^2+1-jh)}{i!} \in \mathbb{Z}$. When p is a prime number and $p^\alpha < h$, then $(p^\alpha, h) = 1$. Therefore among the numbers $h^2+1, h^2+1-h, \dots, h^2+1-\left(\left\lfloor \frac{i}{p^\alpha} \right\rfloor p^\alpha - 1\right)h$ there are $\left\lfloor \frac{i}{p^\alpha} \right\rfloor$, which are divisible by p^α. So

$$p^{\sum_{\alpha=1}^{\infty} \left\lfloor \frac{i}{p^\alpha} \right\rfloor} \mid \prod_{j=1}^{i-1}(h^2+1-jh).$$

By the Legendre's formula, in the prime factorization of $i!$ $(i < h)$ the exponent of p is $\sum_{\alpha=1}^{\infty} \left\lfloor \frac{i}{p^\alpha} \right\rfloor$.

Let

$$A(z) = \sum_{i=1}^{h-1} a_i'(1-z^h)^{i-1} + \frac{h^{h-1}z^{h^2}}{1-z^h} = \sum_{n=0}^{\infty} a_n z^n.$$

Then $a_n = 0$ or $a_n = h^{h-1}$ holds for $n \neq ih$, $0 \leq i \leq h-2$.
Clearly,

$$B(z) = \frac{h^{h-1}z^{h^2+1}}{1-z^h} = \sum_{n=0}^{\infty} b_n z^n,$$

where $b_n = 0$ or $b_n = h^{h-1}$.

It remains to show that the integers $a_0, a_h, a_{2h}, \dots, a_{(h-2)h}$ form primitive residue classes modulo h. If $0 \leq k \leq h-2$ we get that

$$a_{kh} = \sum_{j=k+1}^{h-1} (-1)^k a_j' \binom{j-1}{k}.$$

For $1 \leq j \leq h-2$ we have $h | a_j'$, thus using Wilson's Theorem we get

$$a_{kh} \equiv (-1)^k a_{h-1}' \binom{h-2}{k} \equiv (-1)^k \frac{\prod_{j=0}^{h-2}(h^2+1-jh)}{(h-1)!} \cdot (-1)^{h-1} \frac{(h-2)!}{k!(h-2-k)!} \equiv$$

$$\equiv (-1)^k \cdot \frac{1}{-1} \cdot (-1)^{h-1} \frac{(h-1)!}{(-1)(k+1)!(h-2-k)!} \cdot (k+1) \equiv$$

$$\equiv (-1)^{k+h-1} \frac{(h-1)!(-1)^{k+1}}{(h-1)(h-2)\dots(h-k-1)(h-k-2)!} \cdot (k+1) \equiv$$

$$\equiv (-1)^h \frac{(h-1)!}{(h-1)!}(k+1) \equiv -(k+1) \pmod{h}.$$

This show that statement 2 holds. This completes the proof of Theorem 2. ■

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