A converse to an extension of a theorem of Erdős and Fuchs *

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Abstract

Let $A = \{a_1, a_2, \ldots\}$, $0 \le a_1 \le a_2 \le \ldots$ be an infinite sequence of nonnegative integers, let $h \ge 2$ be a fixed positive integer and denote by $r_h(A,n)$ the number of solutions of $n = a_{i_1} + \ldots + a_{i_h}$. Min Tang proved that $\sum_{n=0}^{N} r_h(A,n) = cN + o(N^{1/4})$ cannot hold for any constant $c > 0$. In this paper we prove the existence of a sequence A satisfying $\sum_{n=0}^{N} r_h(A,n) = cN + O(N^{1-\frac{1.5}{h}})$.

1 Introduction

Let $h \ge 2$ be a fixed integer, let $A = \{a_0, a_1, \ldots\}$ be an infinite, increasing sequence of nonnegative integers and write

$$r_h(A,n) = \#\{(i_1, i_2, \ldots i_h) : i_j \in \mathbb{N}, j = 1, \ldots, h; a_{i_1} + a_{i_2} + \cdots + a_{i_h} = n\}.$$

A theorem of Erdős and Fuchs [1] asserts that $r_2(A,n)$ cannot behave very regularly, namely

$$\sum_{n=0}^{N} r_2(A,n) = cN + o(N^{\frac{1}{4}}(\log N)^{-\frac{1}{2}})$$

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with $c > 0$ is impossible. Jurkat (seemingly unpublished), and later Montgomery and Vaughan [2] improved this theorem to

$$\sum_{n=0}^{N} r_2(A,n) \neq cN + o(N^{\frac{1}{4}}).$$

Vaughan asked whether a further improvement is possible, or in other words, whether there is a sequence such that

$$\sum_{n=0}^{N} r_2(A,n) = cN + O(N^{\frac{1}{4}+\varepsilon}).$$

Ruzsa [3] showed the existence of such a sequence. In fact, he showed that there exists a sequence A of integers such that

$$\sum_{n=0}^{N} r_2(A,n) = cN + O(N^{\frac{1}{4}} \log N)$$

holds.

Already the Erdős-Fuchs theorem has been extended in various directions. Tang [4] showed that $r_h(A,n)$ cannot behave very regularly: for $h > 2$

$$\sum_{n=0}^{N} r_h(A,n) = cN + o(N^{\frac{1}{4}})$$

with $c > 0$ is impossible.

After this result we asked a similar question as Vaughan and we obtain the following result:

Theorem. *For every $h > 2$ there exists a sequence A of integers such that the following equation is true*

$$\sum_{n=0}^{N} r_h(A,n) = N + O\left(N^{1-\frac{1.5}{h}} \log N\right). \tag{1}$$

2 Probabilistic construction of the sequence A

At first we define an infinite increasing sequence $S = \{s_0, s_1 \dots\}$ of nonnegative integers. We will choose the n-th element of A from $\{s_n, s_n+1, \dots, s_{n+1}\}$ by probabilistic method. Denote by E_n the expectation number that how many times occurs the number n in A. We choose these values that

$$\mathbb{E}(r_h(A,n)) = 1 \quad \forall n \in \mathbb{N}$$

hold. If we don't care with the independece, then we get

$$\mathbb{E}(r_h(A,n)) \approx \sum_{\substack{(i_1,\dots,i_h)\\ i_1+\dots+i_h=n}} E_{i_1}E_{i_2}\cdots E_{i_h}$$

So we have to solve the following system of equations:

$$\sum_{\substack{(i_1,\dots,i_h)\\ i_1+\dots+i_h=n}} E_{i_1}E_{i_2}\cdots E_{i_h} = 1 \quad \forall n \in \mathbb{N}$$

Then

$$\left(\sum_{n=0}^{\infty} E_n z^n\right)^h = \sum_{n=0}^{\infty}\left(\sum_{\substack{(i_1,\dots,i_h)\\ i_1+\dots+i_h=n}} E_{i_1}E_{i_2}\cdots E_{i_h}\right) z^n = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

So

$$\sum_{n=0}^{\infty} E_n z^n = (1-z)^{-\frac{1}{h}} = \sum_{n=0}^{\infty}\binom{-\frac{1}{h}}{n}(-z)^n$$

Hence a natural choice of E_n is

$$E_n = (-1)^n\binom{-\frac{1}{h}}{n} = \frac{1}{h}\,\frac{h+1}{2h}\,\frac{2h+1}{3h}\cdots\frac{(n-1)h+1}{nh}$$

Now we can define the sequence S. Denote by s_n the following

$$s_n = \min\left\{m : \sum_{k=0}^{m} E_k > n\right\}.$$

From this we get the random sequence A in the following way:

$$\mathbb{P}(a_n = a) = \begin{cases} \sum_{k=0}^{s_n} E_k - n & \text{if} \quad a = s_n \\ E_a & \text{if} \quad s_n < a < s_{n+1} \\ (n+1) - \sum_{k=0}^{s_{n+1}-1} E_k & \text{if} \quad a = s_{n+1} \\ 0 & \text{otherwise} \end{cases}$$

We will show that with probability 1 this sequence satisfies Theorem.

3 PROOF

The following lemma gives a precise estimation of integer s_n.

Lemma 1. *There exist real numbers $c_1 = c_1(h) > 0$ and $c_2 = c_2(h)$ such that the following equation holds:*

$$s_n = c_1(n - c_2)^h + O(1). \tag{2}$$

Proof. Clearly, $E_0 = 1$ and for $k \geq 1$ we have

$$E_k = \frac{1}{h}\frac{h+1}{2h}\frac{2h+1}{3h}\cdots\frac{(k-1)h+1}{kh} = \exp\left\{\sum_{i=1}^{k}\log\left(1-\frac{h-1}{ih}\right)\right\} =$$
$$= \exp\left\{-\sum_{i=1}^{k}\frac{h-1}{ih} - \frac{1}{2}\sum_{i=1}^{k}\frac{(h-1)^2}{i^2h^2} - \frac{1}{3}\sum_{i=1}^{k}\frac{(h-1)^3}{i^3h^3} - \cdots\right\} =$$
$$= \exp\left\{\frac{h-1}{h}\left(-\log k + C_1 + O\left(\frac{1}{k}\right)\right)\right\} = C_2k^{-\frac{h-1}{h}} + O\left(k^{-2+\frac{1}{h}},\right)$$

that is

$$E_k = C_2k^{-\frac{h-1}{h}} + e_k, \quad \text{where} \quad e_k = O\left(k^{-2+\frac{1}{h}}\right). \tag{3}$$

Using this approximation we obtain

$$\sum_{k=0}^{m}E_k = 1 + \sum_{k=1}^{m}\left(C_2k^{-\frac{h-1}{h}} + e_k\right) = C_2\sum_{k=1}^{m}k^{-\frac{h-1}{h}} + C_3 + O\left(m^{-\frac{h-1}{h}}\right)$$

Obviously,

$$\sum_{k=1}^{m}\frac{1}{k^{\frac{h-1}{h}}} = \int_1^{m+1}\frac{1}{\lfloor x\rfloor^{\frac{h-1}{h}}}\mathrm{d}x = \int_1^{m+1}\frac{1}{x^{\frac{h-1}{h}}}\mathrm{d}x + \int_1^{m+1}\left(\frac{1}{\lfloor x\rfloor^{\frac{h-1}{h}}} - \frac{1}{x^{\frac{h-1}{h}}}\right)\mathrm{d}x =$$
$$= \left[hx^{\frac{1}{h}}\right]_1^{m+1} + \int_1^{\infty}\frac{x^{\frac{h-1}{h}} - \lfloor x\rfloor^{\frac{h-1}{h}}}{x^{\frac{h-1}{h}}\lfloor x\rfloor^{\frac{h-1}{h}}}\mathrm{d}x - \int_{m+1}^{\infty}\frac{x^{\frac{h-1}{h}} - \lfloor x\rfloor^{\frac{h-1}{h}}}{x^{\frac{h-1}{h}}\lfloor x\rfloor^{\frac{h-1}{h}}}\mathrm{d}x = hm^{\frac{1}{h}} + C_4 + O\left(m^{-\frac{h-1}{h}}\right). \tag{4}$$

Hence

$$\sum_{k=0}^{m}E_k = C_2hm^{\frac{1}{h}} + C_5 + O(m^{-\frac{h-1}{h}}).$$

By definition of integer s_n we have $n < \sum_{k=0}^{s_n}E_{s_n} \leq n + E_{s_n}$, therefore $C_2hs_n^{\frac{1}{h}} =$

$n + O(1)$, that is

$$s_n \asymp n^h,$$

therefore

$$\sum_{k=0}^{s_n} E_k = C_2 h s_n^{\frac{1}{h}} + C_5 + O\left(s_n^{-\frac{h-1}{h}}\right) = C_2 h s_n^{\frac{1}{h}} + C_5 + O(n^{-(h-1)}).$$

On the other hand, using again the definition of integer s_n and equality (3) we have

$$n < \sum_{k=0}^{s_n} E_k \leq n + E_{s_n} = n + O\left(n^{-(h-1)}\right),$$

that is

$$\sum_{k=0}^{s_n} E_k = n + O\left(n^{-(h-1)}\right).$$

Our statement follows from the equality

$$C_2 h s_n^{\frac{1}{h}} + C_5 + O\left(n^{-(h-1)}\right) = n + O\left(n^{-(h-1)}\right) \quad \blacksquare$$

We will need the following Bernstein-type inequality of Hoefding [5].

Lemma 2. *Let $X_1, X_2, \dots X_n$ be independent bounded real random variables, $u_i \leq X_i \leq v_i$ and*

$$\sum_{i=1}^{n} (u_i - v_i)^2 \leq D^2$$

Write $S_n = \sum_{i=1}^{n} X_i$. For every $y > 0$ we have

$$\mathbb{P}(S_n - \mathbb{E}(S_n) \geq yD) \leq exp\{-2y^2\}.$$

We will choose a_i, how we write it in section 2. From the definition of sequence a_n we easily get that the following properties are true:

$$\sum_{l=0}^{\infty} \mathbb{P}(a_n = l) = 1 \quad \forall n \in \mathbb{N} \tag{5}$$

and

$$\sum_{n=0}^{\infty} \mathbb{P}(a_n = l) = E_l \quad \forall l \in \mathbb{N} \tag{6}$$

Fix an integer N and define $\delta_{i_1,\ldots,i_h}$ and $d_{i_1,\ldots,i_h}$ in the following way

$$\delta_{i_1,\ldots,i_h} = \begin{cases} 1 & \text{if} \quad a_{i_1} + a_{i_2} + \ldots + a_{i_h} \leq N \\ 0 & \text{if} \quad a_{i_1} + a_{i_2} + \ldots + a_{i_h} > N \end{cases}$$

and

$$d_{i_1,\ldots,i_h} = \sum_{j_1=0}^{\infty} \cdots \sum_{j_h=0}^{N-j_1-\ldots-j_{h-1}} \mathbb{P}(a_{i_1} = j_1)\mathbb{P}(a_{i_2} = j_2)\ldots\mathbb{P}(a_{i_h} = j_h)$$

Then

$$\sum_{(i_1,\ldots,i_h)} \delta_{i_1,\ldots,i_h} = \#\{(i_1,\ldots,i_h) : a_{i_1} + a_{i_2} + \ldots + a_{i_h} \leq N\} = \sum_{n=0}^{N} r_h(A,n)$$

and by (5) and (6)

$$\sum_{(i_1,\ldots,i_h)} d_{i_1,\ldots,i_h} = \sum_{i_1=0}^{\infty} \cdots \sum_{i_h=0}^{\infty} \sum_{j_1=0}^{\infty} \cdots \sum_{j_h=0}^{N-j_1-\ldots-j_{h-1}} \mathbb{P}(a_{i_1} = j_1)\ldots\mathbb{P}(a_{i_h} = j_h) =$$

$$= \sum_{j_1=0}^{\infty} \cdots \sum_{j_h=0}^{N-j_1-\ldots-j_{h-1}} \sum_{i_1=0}^{\infty} \cdots \sum_{i_h=0}^{\infty} \mathbb{P}(a_{i_1} = j_1)\ldots\mathbb{P}(a_{i_h} = j_h) =$$

$$= \sum_{j_1=0}^{\infty} \cdots \sum_{j_h=0}^{N-j_1-\ldots-j_{h-1}} \sum_{i_1=0}^{\infty} \mathbb{P}(a_{i_1} = j_1) \ldots \sum_{i_h=0}^{\infty} \mathbb{P}(a_{i_h} = j_h) =$$

$$= \sum_{j_1=0}^{\infty} \cdots \sum_{j_h=0}^{N-j_1-\ldots-j_{h-1}} E_{j_1}\ldots E_{j_h} = \sum_{\substack{(j_1,\ldots,j_h) \\ j_1+\ldots+j_h \leq N}} E_{j_1}\ldots E_{j_h} = N+1$$

Further

$$\delta_{i_1,\ldots,i_h} = d_{i_1,\ldots,i_h} = \begin{cases} 1 & \text{if} \quad s_{i_1+1} + s_{i_2+1} + \ldots + s_{i_h+1} \leq N \\ 0 & \text{if} \quad s_{i_1} + s_{i_2} + \ldots + s_{i_h} > N \end{cases} \tag{7}$$

So we get the following

$$\sum_{n=0}^{N} r_h(A,n) - (N+1) = \sum_{(i_1,\ldots,i_h)} (\delta_{i_1,\ldots,i_h} - d_{i_1,\ldots,i_h}) =$$

$$= \sum_{\substack{(i_1,\ldots,i_h) \\ s_{i_1+1}+\ldots+s_{i_h+1} \leq N}} (\delta_{i_1,\ldots,i_h} - d_{i_1,\ldots,i_h}) + \sum_{\substack{(i_1,\ldots,i_h) \\ s_{i_1}+\ldots+s_{i_h} \leq N < s_{i_1+1}+\ldots+s_{i_h+1}}} (\delta_{i_1,\ldots,i_h} - d_{i_1,\ldots,i_h}) +$$

$$+ \sum_{\substack{(i_1,\ldots,i_h) \\ s_{i_1}+\ldots+s_{i_h} > N}} (\delta_{i_1,\ldots,i_h} - d_{i_1,\ldots,i_h}) \tag{8}$$

Because of (7) the first and the third sums are equal to 0.

We will show that

$$\sum_{\substack{(i_1,\ldots,i_h)\\ s_{i_1}+\ldots+s_{i_h}\le N<s_{i_1+1}+\ldots+s_{i_h+1}}} (\delta_{i_1,\ldots,i_h}-d_{i_1,\ldots,i_h}) =$$

$$= h! \sum_{\substack{(i_1,\ldots,i_h)\\ 0\le i_1<\ldots<i_h\\ s_{i_1}+\ldots+s_{i_h}\le N<s_{i_1+1}+\ldots+s_{i_h+1}}} (\delta_{i_1,\ldots,i_h}-d_{i_1,\ldots,i_h}) + O\left(N^{1-\frac{2}{h}}\right) \quad (9)$$

In order to prove (9) we have to calculate the number of h-tuples $(i_1,\ldots,i_h)$, where $i_s=i_t$ for some $1\le s<t\le h$. We may suppose without loss of generality that $s=h-1$ and $t=h$. By Lemma 1 for some constants C_6 and C_7 we have

$$\#\{(i_1,\ldots,i_h): s_{i_1}+\ldots+s_{i_h}\le N<s_{i_1+1}+\ldots+s_{i_h+1}, s_{i_{h-1}}=s_{i_h}\} =$$

$$= \#\{(i_1,\ldots,i_{h-1}): s_{i_1}+\ldots+s_{i_{h-2}}+2s_{i_{h-1}}\le N<s_{i_1+1}+\ldots+s_{i_{h-2}+1}+2s_{i_{h-1}+1}\} \le$$

$$\sum_{(i_1,\ldots,i_{h-2})} \#\{i_{h-1}:$$

$$\sum_{j=1}^{h-2} c_1(i_j-c_2)^h+2c_1(i_{h-1}-c_2)^h-C_6 \le N \le \sum_{j=1}^{h-2} c_1(i_j+1-c_2)^h+2c_1(i_{h-1}+1-c_2)^h+C_7\} \le$$

$$\sum_{(i_1,\ldots,i_{h-2})} \#\left\{ i_{h-1}: \sqrt[h]{\frac{N-C_7}{2c_1}-\frac{1}{2}\sum_{j=1}^{h-2}(i_j+1-c_2)^h}+c_2-1 \le i_{h-1} \le\right.$$

$$\left.\sqrt[h]{\frac{N+C_6}{2c_1}-\frac{1}{2}\sum_{j=1}^{h-2}(i_j-c_2)^h}+c_2\right\} \le$$

$$\sum_{(i_1,\ldots,i_{h-2})}\left(\sqrt[h]{\frac{N}{2c_1}-\frac{1}{2}\sum_{j=1}^{h-2}(i_j-c_2)^h}-\sqrt[h]{\frac{N}{2c_1}-\frac{1}{2}\sum_{j=1}^{h-2}(i_j+1-c_2)^h}+O(1)\right) =$$

$$= \sum_{(i_1,\ldots,i_{h-2})}\left(\sqrt[h]{\frac{N}{2c_1}-\frac{1}{2}\sum_{j=1}^{h-2}(i_j-c_2)^h}-\sqrt[h]{\frac{N}{2c_1}-\frac{1}{2}\sum_{j=1}^{h-2}(i_j+1-c_2)^h}\right)+O(N^{\frac{h-2}{h}})$$

The last sum contains telescopic sums. The number of telescopic sums is $O\left(N^{\frac{h-3}{h}}\right)$ and a telescopic sum is $O\left(N^{\frac{1}{h}}\right)$, which proves formula (9).

By the Borel-Cantelli lemma, it is enough to show that for some constant C_8

$$\mathbb{P}\left(\left|\sum_{\substack{(i_1,\dots,i_h)\\ 0\le i_1\dots<i_h\\ s_{i_1}+\dots+s_{i_h}\le N<s_{i_1+1}+\dots+s_{i_h+1}}}(\delta_{i_1,\dots,i_h}-d_{i_1,\dots,i_h})\right|\ge C_8N^{1-\frac{1.5}{h}}\log N\right)=O\left(N^{-\frac{h+2}{h}}\right) \tag{10}$$

To prove this we will decompose the sum in (10) to a sum of independent variables (because the variables $\delta_{i_1,\dots,i_h}$ are not independent).

Let $(i_1,\dots,i_{h-2})$, $0\le i_1<i_2<\dots<i_{h-2}$ be a $(h-2)$-tuple and let

$$I_{i_1,\dots,i_{h-2},i_h}:=\{i_{h-1}:0\le i_1<i_2<\dots<i_{h-1}<i_h,\sum_{j=1}^{h}s_{i_j}\le N\le\sum_{j=1}^{h}s_{i_j+1}\}$$

These $I_{i_1,\dots,i_{h-2},i_h}$'s are intervals. We will show, there exsists an integer K so that if $i_h\equiv i_h^{`}\pmod K$ and $i_h<i_h^{`}$ then $I_{i_1,\dots,i_{h-2},i_h}\cap I_{i_1,\dots,i_{h-2},i_h^{`}}=\emptyset$. In order to prove it let $K$ be an integer and let us suppose that for some $(h-2)$-tuple $(i_1,\dots,i_{h-2})$ there exist integers $i_h$ and $i_h^{`}$ so that $i_h\equiv i_h^{`}\pmod K$, where $i_h<i_h^{`}$ and $I_{i_1,\dots,i_{h-2},i_h}\cap I_{i_1,\dots,i_{h-2},i_h^{`}}\neq\emptyset$. Using the definition of $I_{i_1,\dots,i_{h-2},i_h}$ and $I_{i_1,\dots,i_{h-2},i_h^{`}}$ we get

$$\sum_{j=1}^{h-1}s_{i_j}+s_{i_h}\le N\le\sum_{j=1}^{h-1}s_{i_j+1}+s_{i_h+1}$$

and

$$\sum_{j=1}^{h-1}s_{i_j}+s_{i_h^{`}}\le N\le\sum_{j=1}^{h-1}s_{i_j+1}+s_{i_h^{`}+1}$$

But $s_{i_h+K}\le s_{i_h^{`}}$, therefore

$$s_{i_1}+\dots+s_{i_{h-1}}+s_{i_h+K}\le N\le s_{i_1+1}+\dots+s_{i_{h-1}+1}+s_{i_h+1}$$

By Lemma 1 we get

$$\sum_{j=1}^{h-1}c_1(i_j-c_2)^h+c_1(i_h+K-c_2)^h-C_9\le$$

$$\le\sum_{j=1}^{h-1}c_1(i_j+1-c_2)^h+c_1(i_h+1-c_2)^h+C_{10} \tag{11}$$

To rearrange (10) and estimate

$$c_1h(K-1)(i_h+1-c_2)^{h-1} \le c_1(i_h+K-c_2)^h - c_1(i_h+1-c_2)^h \le$$
$$\le \sum_{j=1}^{h-1} \left(c_1(i_j+1-c_2)^h - c_1(i_j-c_2)^h\right) + C_{11} \le hc_1(i_{h-1}+1-c_2)^{h-1} + C_{12},$$

a contradiction, if K is large enough.

For a $(h-2)$-tuple $(i_1,\ldots,i_{h-2})$ let

$$D^2_{i_1,\ldots,i_{h-2}} = \sum_{i_h} \left|I_{i_1,\ldots,i_{h-2},i_h}\right|^2$$

It is enough to show that

$$\sum_{(i_1,\ldots,i_{h-2})} D^2_{i_1,\ldots,i_{h-2}} = O\left(N^{\frac{h-1}{h}}\log N\right), \tag{12}$$

because by Cauchy-Schwarz inequality we have

$$D = \sum_{(i_1,\ldots,i_{h-2})} D_{i_1,\ldots,i_{h-2}} \le \sqrt{\sum_{(i_1,\ldots,i_{h-2})} D^2_{i_1,\ldots,i_{h-2}} \#\{(i_1,\ldots,i_{h-2})\}} =$$
$$= \sqrt{O\left(N^{\frac{h-1}{h}}\log N\right) O\left(N^{\frac{h-2}{h}}\right)} = O\left(N^{1-\frac{1.5}{h}}\sqrt{\log N}\right)$$

and therefore by Hoeffding's inequality we have

$$\mathbb{P}\left(\left|\sum_{\substack{(i_1,\ldots,i_h)\\ 0\le i_1<\ldots<i_h \\ s_{i_1}+\ldots+s_{i_h}\le N<s_{i_1+1}+\ldots+s_{i_h+1}}} (\delta_{i_1,\ldots,i_h} - d_{i_1,\ldots,i_h})\right| \ge DK\sqrt{2\log N}\right) =$$
$$= \mathbb{P}\left(\left|\sum_{\substack{(i_1,\ldots i_{h-2})\\ 0\le i_1<\ldots<i_{h-2}}} \sum_{\substack{(i_{h-1},i_h)\\ i_{h-2}<i_{h-1}<i_h \\ s_{i_1}+\ldots+s_{i_h}\le N<s_{i_1+1}+\ldots+s_{i_h+1}}} (\delta_{i_1,\ldots,i_h} - d_{i_1,\ldots,i_h})\right| \ge \right.$$
$$\left.\sum_{(i_1,\ldots,i_{h-2})} D_{i_1,\ldots,i_{h-2}} K\sqrt{2\log N}\right) \le$$

$$\leq \sum_{\substack{(i_1,\dots,i_{h-2})\\ 0\leq i_1<\dots<i_{h-2}\\ s_{i_1}+\dots+s_{i_{h-2}}<N}} \sum_{k=0}^{K-1} \mathbb{P}\left(\left| \sum_{\substack{(i_{h-1},i_h)\\ i_{h-2}<i_{h-1}<i_h\\ i_h\equiv k \pmod K\\ s_{i_1}+\dots+s_{i_h}\leq N<s_{i_1+1}+\dots+s_{i_h+1}}} (\delta_{i_1,\dots,i_h} - d_{i_1,\dots,i_h}) \right| \geq D_{i_1,\dots,i_{h-2}}\sqrt{2\log N} \right)$$

$$\leq \sum_{\substack{(i_1,\dots,i_{h-2})\\ 0\leq i_1<\dots<i_{h-2}\\ s_{i_1}+\dots+s_{i_{h-2}}<N}} \sum_{k=0}^{K-1} \exp\{-2\log N\} = O\left(N^{-1-\frac{2}{h}}\right)$$

and we may use the Borel-Cantelli Lemma.

In order to prove (12) let $(i_1,\dots,i_{h-2})$, $0\leq i_1<\dots<i_{h-2}$ be a fixed $(h-2)$-tuple. Recall the definition

$$I_{i_1,\dots,i_{h-2},i_h} = \{i_{h-1} : s_{i_1}+\dots+s_{i_h} \leq N < s_{i_1+1}+\dots+s_{i_h+1}\}.$$

Denote by k' the maximum value of i_h. Let

$$H = H_{i_1,\dots,i_h} = \frac{N}{c_1} - \sum_{j=1}^{h-2}(i_j-c_2)^h$$

It is easy to see that

$$H \asymp N, \tag{13}$$

and $H>0$ if N is large enough.

To prove Theorem it is enough to verify

Lemma 3. *1.* $k' = \sqrt[h]{H-(i_{h-2}-c_2)^h} + O(1)$

2. $|I_{i_1,\dots,i_{h-2},i_h}| = O(N^{\frac{h-1}{h^2}})$ *for every* $(h-2)$*-tuple* $(i_1,\dots,i_{h-2})$

3. $|I_{i_1,\dots,i_{h-2},i_h}| = O\left(\frac{N^{\frac{h-1}{h}}}{(H-(i_h-c_2)^h)^{\frac{h-1}{h}}}\right)$, *if* $H-(i_h-c_2)^h)\geq 1$.

because if C is a fixed constant and N is large enough then

$$\sum_{(i_1,\dots,i_{h-2})} D^2_{i_1,\dots,i_{h-2}} = \sum_{\substack{(i_1,\dots,i_{h-2})\\ i_{h-2}\leq CN^{\frac{h-1}{h^2}}}} D^2_{i_1,\dots,i_{h-2}} + \sum_{\substack{(i_1,\dots,i_{h-2})\\ i_{h-2}>CN^{\frac{h-1}{h^2}}}} D^2_{i_1,\dots,i_{h-2}},$$

where for $i_{h-2}\leq CN^{\frac{h-1}{h^2}}$ by Lemma 2 and (13) we have

$$D^2_{i_1,\ldots,i_{h-2}} = \sum_{i_h} |I_{i_1,\ldots,i_{h-2},i_h}|^2 = \sum_{i_h > \sqrt[h]{H}+c_2-1} |I_{i_1,\ldots,i_{h-2},i_h}|^2 + \sum_{i_h \le \sqrt[h]{H}+c_2-1} |I_{i_1,\ldots,i_{h-2},i_h}|^2 =$$

$$O(N^{\frac{2(h-1)}{h^2}}) + \sum_{r\ge 1} |I_{i_1,\ldots,i_{h-2},\sqrt[h]{H}+c_2-r}|^2 =$$

$$O(N^{\frac{2(h-1)}{h^2}}) + \sum_{r\ge 1} \left(\frac{O(N^{\frac{2(h-1)}{h}})}{((\sqrt[h]{H})^h - (\sqrt[h]{H}+c_2-r-c_2)^h)^{\frac{2(h-1)}{h}}} \right) =$$

$$O(N^{\frac{2(h-1)}{h^2}}) + O(N^{\frac{2(h-1)}{h}} H^{-\frac{2(h-1)^2}{h^2}}) \sum_r \frac{1}{r^{\frac{2(h-1)}{h}}} = O(N^{\frac{2(h-1)}{h^2}}), \quad (14)$$

therefore

$$\sum_{\substack{(i_1,\ldots,i_{h-2}) \\ i_{h-2} \le CN^{\frac{h-1}{h^2}}}} D^2_{i_1,\ldots,i_{h-2}} = O(N^{\frac{(h-2)(h-1)}{h^2}}) O(N^{\frac{2(h-1)}{h^2}}) = O(N^{\frac{h-1}{h}})$$

If $i_{h-2} > CN^{\frac{h-1}{h^2}}$, where C is a large fixed constant and N is large enough, then

$$H - (i_h - c_2)^h \ge 1,$$

because

$$c_1 \sum_{j=1}^{h} (i_j - c_2)^h - C_{13} \le N,$$

that is

$$1 \le (i_{h-2} - c_2)^h - \frac{C_{13}}{c_1} \le (i_{h-1} - c_2)^h - \frac{C_{13}}{c_1} \le H - (i_h - c_2)^h$$

thus by Lemma 2 we have

$$k' = \sqrt[h]{H - (i_{h-2} - c_2)^h} + O(1) = \sqrt[h]{H} \sqrt[h]{1 - \frac{(i_{h-2} - c_2)^h}{H}} + O(1)$$
$$\le \sqrt[h]{H} \left(1 - \frac{(i_{h-2} - c_2)^h}{hH} \right) + O(1) \le \sqrt[h]{H} - \frac{(i_{h-2} - c_2)^h}{hH^{\frac{h-1}{h}}} + O(1) \le$$
$$\sqrt[h]{H} + c_2 - \frac{i_{h-2}^h}{2hH^{\frac{h-1}{h}}}$$

Hence

$$\sum_{\substack{(i_1,\dots,i_{h-2})\\ 0\le i_1<\dots<i_{h-2}\\ i_{h-2}>CN^{\frac{h-1}{h^2}}}} D^2_{i_1,\dots,i_{h-2}} = \sum_{\substack{i_{h-2}\\ i_{h-2}>CN^{\frac{h-1}{h^2}}}} \sum_{\substack{(i_1,\dots,i_{h-3})\\ 0\le i_1<\dots<i_{h-3}\\ i_{h-3}<i_{h-2}}} D^2_{i_1,\dots,i_{h-2}} =$$

$$\sum_{\substack{i_{h-2}\\ i_{h-2}>CN^{\frac{h-1}{h^2}}}} \sum_{\substack{(i_1,\dots,i_{h-3})\\ 0\le i_1<\dots<i_{h-3}\\ i_{h-3}<i_{h-2}}} \sum_{i_h} |I_{i_1,\dots,i_{h-2},i_h}|^2 =$$

$$\sum_{\substack{i_{h-2}\\ i_{h-2}>CN^{\frac{h-1}{h^2}}}} \sum_{\substack{(i_1,\dots,i_{h-3})\\ 0\le i_1<\dots<i_{h-3}\\ i_{h-3}<i_{h-2}}} \sum_{i_h} O\left(\frac{N^{\frac{2(h-1)}{h}}}{(H-(i_h-c_2)^h)^{\frac{2(h-1)}{h}}}\right) \le$$

$$\sum_{\substack{i_{h-2}\\ i_{h-2}>CN^{\frac{h-1}{h^2}}}} \sum_{\substack{(i_1,\dots,i_{h-3})\\ 0\le i_1<\dots<i_{h-3}\\ i_{h-3}<i_{h-2}}} \sum_{r\ge\frac{i_{h-2}^h}{2hH^{\frac{h-1}{h}}}} O\left(\frac{N^{\frac{2(h-1)}{h}}}{(H-(\sqrt[h]{H}+c_2-r-c_2)^h)^{\frac{2(h-1)}{h}}}\right) \le$$

$$\sum_{\substack{i_{h-2}\\ i_{h-2}>CN^{\frac{h-1}{h^2}}}} \sum_{\substack{(i_1,\dots,i_{h-3})\\ 0\le i_1<\dots<i_{h-3}\\ i_{h-3}<i_{h-2}}} N^{\frac{2(h-1)}{h}} H^{-\frac{2(h-1)^2}{h^2}} \sum_{r\ge\frac{i_{h-2}^h}{2hH^{\frac{h-1}{h}}}} O\left(r^{-\frac{2(h-1)}{h}}\right)$$

where

$$\sum_{r\ge\frac{i_{h-2}^h}{2hH^{\frac{h-1}{h}}}} r^{-\frac{2(h-1)}{h}} = O\left(\int_{\frac{i_{h-2}^h}{2hH^{\frac{h-1}{h}}}}^{\infty} x^{-\frac{2(h-1)}{h}}\mathrm{d}x\right) = O\left(\frac{i_{h-2}^{2-h}}{N^{\frac{(h-1)(h-2)}{h^2}}}\right)$$

Therefore

$$\sum_{\substack{(i_1,\dots,i_{h-2})\\ 0\le i_1<\dots<i_{h-2}\\ i_{h-2}>CN^{\frac{h-1}{h^2}}}} D^2_{i_1,\dots,i_{h-2}} = O\left(\sum_{\substack{i_{h-2}\\ i_{h-2}>CN^{\frac{h-1}{h^2}}}} \sum_{\substack{(i_1,\dots,i_{h-3})\\ 0\le i_1<\dots<i_{h-3}\\ i_{h-3}<i_{h-2}}} N^{\frac{2(h-1)}{h}} N^{-\frac{2(h-1)^2}{h^2}} i_{h-2}^{2-h} N^{\frac{(h-1)(h-2)}{h^2}}\right)$$

The inner sum has $O(i_{h-2}^{h-3})$ member, therefore

$$\sum_{\substack{(i_1,\dots,i_h)\\ 0\le i_1<\dots<i_{h-2}\\ i_{h-2}>CN^{\frac{h-1}{h^2}}}} D^2_{i_1,\dots,i_{h-2}} = O\left(\sum_{\substack{i_{h-2}\\ i_{h-2}>CN^{\frac{h-1}{h^2}}}} N^{\frac{h-1}{h}} i_{h-2}^{-1}\right) = O\left(N^{\frac{h-1}{h}}\log N\right).$$

Proof of Lemma 3 (1): It is easy to see that the maximum value is at

$$k' = \max\{l : s_{i_1} + \ldots + s_{i_{h-2}} + s_{i_{h-2}+1} + s_l \le N \le s_{i_1+1} + \ldots + s_{i_{h-2}+1} + s_{i_{h-2}+2} + s_{l+1}\}$$

and using Lemma 1 a routine calculation gives the statement.

(2), (3): By Lemma 1 there exist constants C_{14} and C_{15} such that

$$|I_{i_1,\ldots,i_{h-2},i_h}| = \#\{i_{h-1} : s_{i_1} + \ldots + s_{i_h} \le N \le s_{i_1+1} + \ldots + s_{i_h+1}\} \le \#\{i_{h-1} : c_1(i_{h-1} + 1 - c_2)^h \ge N - \sum_{j=1}^{h-2} c_1(i_j + 1 - c_2)^h - c_1(i_h + 1 - c_2)^h - C_{14}; c_1(i_{h-1} - c_2)^h \le N - \sum_{j=1}^{h-2} c_1(i_j - c_2)^h - c_1(i_h - c_2)^h + C_{15}\},$$

therefore for some constant C_{16}

$$H - (i_h - c_2)^h - C_{16}N^{\frac{h-1}{h}} \le (i_{h-1} - c_2)^h \le H - (i_h - c_2)^h + \frac{C_{15}}{c_1}, \qquad (15)$$

which implies part (2).

To prove part (3) we distinguish two cases: If

$$\frac{N}{c_1} - \sum_{j=1}^{h-2}(i_j + 1 - c_2)^h - (i_h + 1 - c_2)^h < 0,$$

then

$$H - (i_h - c_2)^h = O(N^{\frac{h-1}{h}}),$$

thus by (16)

$$|I_{i_1,\ldots,i_{h-2},i_h}| = O(\sqrt[h]{H - (i_h - c_2)^h}) = O\left(\frac{N^{\frac{h-1}{h}}}{(H - (i_h - c_2)^h)^{\frac{h-1}{h}}}\right)$$

If

$$\frac{N}{c_1} - \sum_{j=1}^{h-2}(i_j + 1 - c_2)^h - (i_h + 1 - c_2)^h \ge 0,$$

then

$$|I_{i_1,\ldots,i_{h-2},i_h}| \leq$$

$$\leq \sqrt[h]{H-(i_h-c_2)^h} - \sqrt[h]{\frac{N}{c_1} - \sum_{j=1}^{h-2}(i_j+1-c_2)^h - c_1(i_h+1-c_2)^h} + O(1)$$

$$\leq \frac{H-(i_h-c_2)^h - (\frac{N}{c_1} - \sum_{j=1}^{h-2}(i_j+1-c_2)^h - (i_h+1-c_2)^h)}{(H-(i_h-c_2)^h)^{\frac{h-1}{h}}} + O(1) \leq$$

$$\leq \frac{\sum_{j=1}^{h-2}\left((i_j+1-c_2)^h - (i_j-c_2)^h\right) + (i_h+1-c_2)^h - (i_h-c_2)^h}{(H-(i_h-c_2)^h)^{\frac{h-1}{h}}} + O(1) =$$

$$O\left(\frac{N^{\frac{h-1}{h}}}{(H-(i_h-c_2)^h)^{\frac{h-1}{h}}}\right)$$

■

References

[1] P. Erdős and W.H.J. Fuchs, On a problem of additive number theory, J. London Math. Soc.,31, (1956), 67-73.

[2] H.L. Montgomery and R. C. Vaughan, On the Erdős-Fuchs theorem, in "A Tribute to Paul Erdős" (A. Baker, B. Bollobás, and A. Hajnal, Eds.), pp. 331-338, Cambridge Univ. Press, Cambridge,1990.

[3] Imre Z. Ruzsa, A converse to a theorem of Erdős and Fuchs, J. of Number Theory, 62, (1997), 397-402.

[4] Min Tang, On a generalization of a theorem of Erdős and Fuchs, Discrete Mathematics, 309, (2009),6288-6293.

[5] W. Hoefding, Probability inequalities for sums of bounded random variables, J. Amer. Stat. Assoc.,58, (1963), 13-30