Sets with almost coinciding representation functions

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Abstract

For a given integer n and a set $\mathcal{S} \subseteq \mathbb{N}$ denote by $R^{(1)}_{h,\mathcal{S}}(n)$ the number of solutions of the equation $n = s_{i_1} + \dots + s_{i_h}$, $s_{i_j} \in \mathcal{S}$, $j = 1, \dots, h$. In this paper we determine all pairs $(\mathcal{A}, \mathcal{B})$, $\mathcal{A}, \mathcal{B} \subseteq \mathbb{N}$ for which $R^{(1)}_{3,\mathcal{A}}(n) = R^{(1)}_{3,\mathcal{B}}(n)$ from a certain point on. We discuss some related problems.

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1 Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. For a given infinite set $\mathcal{A} \subset \mathbb{N}$ the representation functions $R^{(1)}_{h,\mathcal{A}}(n)$, $R^{(2)}_{h,\mathcal{A}}(n)$ and $R^{(3)}_{h,\mathcal{A}}(n)$ are defined in the following way:

$$R^{(1)}_{h,\mathcal{A}}(n) = \#\{(a_{i_1}, \dots, a_{i_h}) : a_{i_1} + \dots + a_{i_h} = n, a_{i_1}, \dots, a_{i_h} \in \mathcal{A}\},$$

$$R^{(2)}_{h,\mathcal{A}}(n) = \#\{(a_{i_1}, \dots, a_{i_h}) : a_{i_1} + \dots + a_{i_h} = n, a_{i_1}, \dots, a_{i_h} \in \mathcal{A}, a_{i_1} \le \dots \le a_{i_h}\},$$

$$R^{(3)}_{h,\mathcal{A}}(n) = \#\{(a_{i_1}, \dots, a_{i_h}) : a_{i_1} + \dots + a_{i_h} = n, a_{i_1}, \dots, a_{i_h} \in \mathcal{A}, a_{i_1} < \dots < a_{i_h}\}.$$

Representation functions have been extensively studied by many authors and are still a fruitful area of research in additive number theory. Using generating functions, Nathanson [6] proved the following result.
Let A, B and T be finite sets of integers. If each residue class modulo m contains exactly the same number of elements of A as elements of B, then we write $A \equiv B \pmod{m}$. If the number of solutions of the congruence $a + t \equiv n \pmod{m}$ with $a \in A$, $t \in T$, equals the number of solutions of the congruence $b + t \equiv n \pmod{m}$ with $b \in B$, $t \in T$ for each residue class n modulo m then we write $A + T \equiv B + T \pmod{m}$.

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Nathanson's Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be infinite sets of nonnegative integers, $\mathcal{A} \neq \mathcal{B}$. Then $R^{(1)}_{2,\mathcal{A}}(n) = R^{(1)}_{2,\mathcal{B}}(n)$ from a certain point on if and only if there exist positive integers N, m and finite sets A, B, T with $A \cup B \subset \{0, 1, \dots, N\}$ and $T \subset \{0, 1, \dots, m-1\}$ such that $A + T \equiv B + T \pmod{m}$, and $\mathcal{A} = A \cup \mathcal{C}$ and $\mathcal{B} = B \cup \mathcal{C}$, where $\mathcal{C} = \{c > N | c \equiv t \pmod{m} \quad \text{for some} \quad t \in T\}$.*

It is clear that $R^{(2)}_{2,\mathcal{A}}(n) = \left\lceil \frac{R^{(1)}_{2,\mathcal{A}}(n)}{2} \right\rceil$ and $R^{(3)}_{2,\mathcal{A}}(n) = \left\lfloor \frac{R^{(1)}_{2,\mathcal{A}}(n)}{2} \right\rfloor$, thus for the sets $\mathcal{A}, \mathcal{B}$ in Nathanson's Theorem we have $R^{(2)}_{2,\mathcal{A}}(n) = R^{(2)}_{2,\mathcal{B}}(n)$ and $R^{(3)}_{2,\mathcal{A}}(n) = R^{(3)}_{2,\mathcal{B}}(n)$ from a certain point on. It is easy to see that the symmetric difference of the sets $\mathcal{A}$ and $\mathcal{B}$ in the above theorem is finite. A. Sárközy asked whether there exist two infinite sets of nonnegative integers $\mathcal{A}$ and $\mathcal{B}$ with infinite symmetric difference, i.e.

$$|(A \cup B) \setminus (A \cap B)| = \infty$$

and

$$R^{(i)}_{2,\mathcal{A}}(n) = R^{(i)}_{2,\mathcal{B}}(n)$$

if $n \geq n_0$, for i = 1, 2, 3. For $i = 1$ the answer is negative (see in [3]). For $i = 2$ G. Dombi [3] and for $i = 3$ Y. G. Chen and B. Wang [2] proved that the set of nonnegative integers can be partitioned into two subsets $\mathcal{A}$ and $\mathcal{B}$ such that $R^{(i)}_{2,\mathcal{A}}(n) = R^{(i)}_{2,\mathcal{B}}(n)$ for all $n \geq n_0$. In [5] Lev gave a common proof to the above mentioned results of Dombi [3] and Chen and Wang [2]. Using generating functions Cs. Sándor [7] determined the sets $\mathcal{A} \subset \mathbb{N}$ for which either

$$R^{(2)}_{2,\mathcal{A}}(n) = R^{(2)}_{2,\mathbb{N}\setminus\mathcal{A}}(n) \quad \text{for all} \quad n \geq n_0$$

or

$$R^{(3)}_{2,\mathcal{A}}(n) = R^{(3)}_{2,\mathbb{N}\setminus\mathcal{A}}(n) \quad \text{for all} \quad n \geq n_0.$$

In [8] M. Tang gave an elementary proof of Cs. Sándor's results and in [1] Y. G. Chen and M. Tang studied related questions. We can rewrite Nathanson's Theorem in equivalent form:

Equivalent form of Nathanson's Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be infinite sets of nonnegative integers, $\mathcal{A} \neq \mathcal{B}$. Then $R^{(1)}_{2,\mathcal{A}}(n) = R^{(1)}_{2,\mathcal{B}}(n)$ from a certain point on if and only if there exist positive integers n_0, M and finite sets $F_{\mathcal{A}}$, $F_{\mathcal{B}}$, T with $F_{\mathcal{A}} \cup F_{\mathcal{B}} \subset \{0, 1, \dots, Mn_0 - 1\}$ and $T \subset \{0, 1, \dots, M-1\}$ such that*

$$\mathcal{A} = F_{\mathcal{A}} \cup \{kM + t : k \geq n_0, t \in T\},$$
$$\mathcal{B} = F_{\mathcal{B}} \cup \{kM + t : k \geq n_0, t \in T\},$$

and

$$(1 - z^M) | (F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)) \, T(z).$$

We conjecture that Nathanson's theorem can be generalized in the following way.

Conjecture. *Let $h \geq 2$, $\mathcal{A}$ and $\mathcal{B}$ be infinite sets of nonnegative integers, $\mathcal{A} \neq \mathcal{B}$. Then $R^{(1)}_{h,\mathcal{A}}(n) = R^{(1)}_{h,\mathcal{B}}(n)$ from a certain point on if and only if there exist positive integers n_0,*

M and sets $F_{\mathcal{A}}$, $F_{\mathcal{B}}$ and T such that $F_{\mathcal{A}} \cup F_{\mathcal{B}} \subset \{0, 1, \ldots, Mn_0 - 1\}$, $T \subset \{0, 1. \ldots, M - 1\}$,

$$\mathcal{A} = F_{\mathcal{A}} \cup \{kM + t : k \geq n_0, t \in T\},$$
$$\mathcal{B} = F_{\mathcal{B}} \cup \{kM + t : k \geq n_0, t \in T\},$$

and

$$(1 - z^M)^{h-1} | \, (F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)) \, T(z)^{h-1}.$$

The next theorem proves the sufficiency of Conjecture.

Theorem 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be infinite sets of nonnegative integers, $\mathcal{A} \neq \mathcal{B}$. If there exist positive integers n_0, M and finite sets $F_{\mathcal{A}}$, $F_{\mathcal{B}}$ and T with $F_{\mathcal{A}} \cup F_{\mathcal{B}} \subset \{0, 1, \ldots, Mn_0 - 1\}$, $T \subset \{0, 1. \ldots, M - 1\}$ such that*

$$\mathcal{A} = F_{\mathcal{A}} \cup \{kM + t : k \geq n_0, t \in T\},$$
$$\mathcal{B} = F_{\mathcal{B}} \cup \{kM + t : k \geq n_0, t \in T\},$$

and

$$(1 - z^M)^{h-1} | \, (F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)) \, T(z)^{h-1}$$

then $R^{(1)}_{h,\mathcal{A}}(n) = R^{(1)}_{h,\mathcal{B}}(n)$ from a certain point on.

We can only prove the above conjecture in the case $h = 3$.

Theorem 2. *Let $\mathcal{A}$ and $\mathcal{B}$ be infinite sets of nonnegative integers, $\mathcal{A} \neq \mathcal{B}$. Then $R^{(1)}_{3,\mathcal{A}}(n) = R^{(1)}_{3,\mathcal{B}}(n)$ from a certain point on if and only if there exist positive integers n_0, M and sets $F_{\mathcal{A}}$, $F_{\mathcal{B}}$ and T with $F_{\mathcal{A}} \cup F_{\mathcal{B}} \subset \{0, 1, \ldots, Mn_0 - 1\}$, $T \subset \{0, 1. \ldots, M - 1\}$ such that*

$$\mathcal{A} = F_{\mathcal{A}} \cup \{kM + t : k \geq n_0, t \in T\}, \tag{1}$$
$$\mathcal{B} = F_{\mathcal{B}} \cup \{kM + t : k \geq n_0, t \in T\}, \tag{2}$$

and

$$(1 - z^M)^2 | \, (F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)) \, T(z)^2. \tag{3}$$

In 2011, Yang [9] gave another proof of Nathanson's theorem without using generating functions. In his paper he posed the following problem.

Problem. *If $p \geq 3$ is a prime and $\mathcal{A}$ is an infinite set of nonnegative integers, then does there exist an infinite set of nonnegative integers $\mathcal{B}$ with $\mathcal{A} \neq \mathcal{B}$ such that $R^{(1)}_{p,\mathcal{A}}(n) = R^{(1)}_{p,\mathcal{B}}(n)$ for all sufficiently large n?*

In this paper we show that the answer of Yang's question is negative.

Theorem 3. *For every prime p there exists an infinite set of nonnegative integers $\mathcal{A}$ such that for any infinite set of integers $\mathcal{B}$, $\mathcal{A} \neq \mathcal{B}$, we have $R^{(1)}_{p,\mathcal{A}}(n) \neq R^{(1)}_{p,\mathcal{B}}(n)$ for infinitely many positive integer n.*

We studied some similar problems and get the following results.

Theorem 4. *For every positive integer $H \geq 2$ there exist infinite sets of nonnegative integers $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A} \neq \mathcal{B}$ such that $R^{(l)}_{h,\mathcal{A}}(n) = R^{(l)}_{h,\mathcal{B}}(n)$, for every $l = 1, 2, 3$ and $2 \leq h \leq H$ from a certain point on.*

In the special case $l = 1$, Theorem 4 cannot be extended for infinitely many h.

Theorem 5. *If for some infinite sets of nonnegative integers $\mathcal{A}$ and $\mathcal{B}$ the representation function $R^{(1)}_{h,\mathcal{A}}(n) = R^{(1)}_{h,\mathcal{B}}(n)$, for $n \geq n_0(h)$, for infinitely many positive integer $h \geq 2$, then $\mathcal{A} = \mathcal{B}$.*

In this paper let $A(z)$, $B(z)$, $F_{\mathcal{A}}(z)$, $F_{\mathcal{B}}$, $T(z)$, $S(z)$ denote the generating functions of the sets $\mathcal{A}$, $\mathcal{B}$, $F_{\mathcal{A}}$, $F_{\mathcal{B}}$, T and $S \subseteq \mathbb{N}$ (i.e. $A(z) = \sum_{a \in \mathcal{A}} z^a$, where z is a complex number, $z = r \cdot e^{2\pi i \theta}$, and these functions converge in the open unit disc).

2 Proof of Theorem 1.

In order to prove Theorem 1 we need to show that $A(z)^h - B(z)^h = P(z)$, where $P(z)$ is a polynomial. By definition of $\mathcal{A}$ and $\mathcal{B}$ we have

$$A(z) = F_{\mathcal{A}}(z) + \frac{z^{n_0 M} T(z)}{1 - z^M}$$

and

$$B(z) = F_{\mathcal{B}}(z) + \frac{z^{n_0 M} T(z)}{1 - z^M}.$$

Therefore using the binomial theorem we get that

$$A(z)^h - B(z)^h = \left(F_{\mathcal{A}}(z) + \frac{z^{n_0 M} T(z)}{1 - z^M} \right)^h - \left(F_{\mathcal{B}}(z) + \frac{z^{n_0 M} T(z)}{1 - z^M} \right)^h =$$

$$= \sum_{k=1}^{h} \binom{h}{k} \left(\frac{z^{n_0 M} T(z)}{1 - z^M} \right)^{h-k} \left(F_{\mathcal{A}}(z)^k - F_{\mathcal{B}}(z)^k \right).$$

Now we verify that for $1 \leq k \leq h-1$ we have

$$(1 - z^M)^{h-k} \left| T(z)^{h-k} \left(F_{\mathcal{A}}(z)^k - F_{\mathcal{B}}(z)^k \right) \right. .$$

Since

$$F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z) | F_{\mathcal{A}}(z)^k - F_{\mathcal{B}}(z)^k,$$

it is enough to show that

$$(1 - z^M)^{h-k} | T(z)^{h-k} (F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)).$$

For a given integer m, $m|M$ denote by $\Phi_m(z)$ the mth cyclomatic polynomial. It remains to prove that

$$\Phi_m(z)^{h-k} | T(z)^{h-k} \left(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z) \right).$$

Let $T(z)=\Phi_m(z)^{k_1}u(z)$ and $F_{\mathcal{A}}(z)-F_{\mathcal{B}}(z)=\Phi_m(z)^{k_2}v(z)$, where $u(z)$ and $v(z)$ are polynomials with property $\Phi_m(z)\not| u(z)v(z)$. By assumption of Theorem 1 we know that $(h-1)k_1+k_2\geq h-1$. Thus either $k_1=0$, then $k_2\geq h-1$, therefore

$$\Phi_m(z)^{h-k}|F_{\mathcal{A}}(z)-F_{\mathcal{B}}(z),$$

or $k_1\geq 1$ and therefore

$$\Phi_m(z)^{h-k}|T(z)^{h-k},$$

which completes the proof. ■

3 Proof of Theorem 2.

First we would like to prove, that if $R^{(1)}_{3,\mathcal{A}}(n)=R^{(1)}_{3,\mathcal{B}}(n)$ from a certain point on then we have nonnegative integers n_0, M and finite sets of nonnegative integers $F_{\mathcal{A}}$, $F_{\mathcal{B}}$, T with $F_{\mathcal{A}}\cup F_{\mathcal{B}}\subset\{0,1,\dots,Mn_0-1\}$, $T\subset\{0,1.\dots,M-1\}$ such that (1), (2) and (3) hold. It is easy to see that there exists a positive integer N_0 such that $\mathcal{A}\cap[N_0,+\infty)=\mathcal{B}\cap[N_0,+\infty)$, because $R^{(1)}_{3,\mathcal{A}}(n)\equiv 0 \pmod 3$ if $\frac{n}{3}\notin\mathcal{A}$, and $R^{(1)}_{3,\mathcal{A}}(n)\equiv 1 \pmod 3$ if $\frac{n}{3}\in\mathcal{A}$. Similarly $R^{(1)}_{3,\mathcal{B}}(n)\equiv 0 \pmod 3$ if $\frac{n}{3}\notin\mathcal{B}$, and $R^{(1)}_{3,\mathcal{B}}(n)\equiv 1 \pmod 3$ if $\frac{n}{3}\in\mathcal{B}$. Thus there exists an integer N_1, finite sets of nonnegative integers $F_{\mathcal{A}}$, $F_{\mathcal{B}}$ and an infinite set of nonnegative integers S with $F_{\mathcal{A}},F_{\mathcal{B}}\subset\{0,1,\dots,N_1\}$, $S\subset\{N_1+1,N_1+2\dots\}$ such that

$$\mathcal{A}=F_{\mathcal{A}}\cup S \tag{4}$$

and

$$\mathcal{B}=F_{\mathcal{B}}\cup S. \tag{5}$$

Since $A(z)$ and $B(z)$ are the generating functions of the sets $\mathcal{A}$ and $\mathcal{B}$, we have

$$A^3(z)=\sum_{n=0}^{\infty}R^{(1)}_{3,\mathcal{A}}(n)z^n,$$

and

$$B^3(z)=\sum_{n=0}^{\infty}R^{(1)}_{3,\mathcal{B}}(n)z^n.$$

Since $R^{(1)}_{3,\mathcal{A}}(n)=R^{(1)}_{3,\mathcal{B}}(n)$, for $n\geq N_2$, it is clear that there is a polynomial $Q(z)$ such that

$$\sum_{n=1}^{\infty}R^{(1)}_{3,\mathcal{A}}(n)z^n-\sum_{n=1}^{\infty}R^{(1)}_{3,\mathcal{B}}(n)z^n=Q(z). \tag{6}$$

Thus we have $A^3(z)-B^3(z)=Q(z)$. In view of (4) and (5) it follows that

$$A(z)=F_{\mathcal{A}}(z)+S(z)$$

and

$$B(z)=F_{\mathcal{B}}(z)+S(z).$$

Hence

$$(S(z)+F_{\mathcal{A}}(z))^3-(S(z)+F_{\mathcal{B}}(z))^3=$$

$$=3S^2(z)F_{\mathcal{A}}(z)+3S(z)F_{\mathcal{A}}^2(z)-3S^2(z)F_{\mathcal{B}}(z)-3S(z)F_{\mathcal{B}}^2(z)+F_{\mathcal{A}}^3(z)-F_{\mathcal{B}}^3(z)=Q(z).$$

Since $F_{\mathcal{A}}$ and $F_{\mathcal{B}}$ are finite sets there is a polynomial $P(z)$ such that

$$3S(z)\left(S(z)+F_{\mathcal{A}}(z)+F_{\mathcal{B}}(z)\right)\left(F_{\mathcal{A}}(z)-F_{\mathcal{B}}(z)\right)=P(z).$$

It follows that there are relatively prime polynomials $P_1(z)$ and $P_2(z)$ such that

$$3S(z)(S(z)+F_{\mathcal{A}}(z)+F_{\mathcal{B}}(z))=\frac{P(z)}{F_{\mathcal{A}}(z)-F_{\mathcal{B}}(z)}=\frac{P_1(z)}{P_2(z)}. \qquad (7)$$

The left hand side of (7) converges in the open unit disc. Then

$$F_{\mathcal{A}}(z)-F_{\mathcal{B}}(z)=z^l(c_0+c_1z+...+c_qz^q),$$

where $|c_0|=1$ and $|c_q|=1$. Thus

$$P_2(z)=z^k(d_0+d_1z+...+d_wz^w),$$

where $|d_0|=1$ and $|d_w|=1$. Assume that $k\neq 0$. Then the right hand side of (7) tends to infinity in absolute value and the left hand side of (7) converges in absolute value when $z\to 0$, which is absurd. So we get that $k=0$. Thus we have

$$P_2(z)=d_0+d_1z+...+d_wz^w,$$

and

$$F_{\mathcal{A}}(z)-F_{\mathcal{B}}(z)=\sum_{n=0}^{N_1}f_nz^n,$$

where all the f_n's are integers and $|f_n|\leq 1$.
We prove the following lemma:

Lemma 1. *If for some complex number z_0, $P_2(z_0)=0$, then $|z_0|\geq 1$.*

Proof. We prove by contradiction. Assume that there exists $z_0\in\mathbb{C}$ such that $P_2(z_0)=0$ and $|z_0|<1$. Take the limit $z\to z_0$ in (7). Then

$$3S(z)\left(S(z)+F_{\mathcal{A}}(z)+F_{\mathcal{B}}(z)\right)\to 3S(z_0)\left(S(z_0)+F_{\mathcal{A}}(z_0)+F_{\mathcal{B}}(z_0)\right),$$

and

$$|3S(z)\left(S(z)+F_{\mathcal{A}}(z)+F_{\mathcal{B}}(z)\right)|\to|3S(z_0)\left(S(z_0)+F_{\mathcal{A}}(z_0)+F_{\mathcal{B}}(z_0)\right)|\in\mathbb{R}.$$

Since $P_1(z)$ and $P_2(z)$ are relatively prime, $P_1(z_0)\neq 0$, we have

$$\left|\frac{P_1(z)}{P_2(z)}\right|\to\infty,$$

as $z\to z_0$, which is absurd. ■

We may suppose that $d_w = 1$. This means that the roots of $P_2(z)$ are algebraic integers. In this case the products of the roots of the polynomial $P_2(z)$ is d_0 and $|d_0| = 1$. It follows from Lemma 1, that the absolut value of each root is 1. Since $d_w = 1$ it is well-known that the roots lies with their conjugates in the closed unit disc. It follows from a well-known theorem of Kronecker [4] that every root is a root of unity. Thus we obtain that

$$P_2(z) = \prod_{j=1}^{u} (z - \varepsilon_j)^{m_j},$$

where ε_j is a root of unity and has the multiplicity m_j.
We prove that for every j, $m_j \le 2$. Assume that there exists an $m_j \ge 3$. Then from (7) we have

$$3S(z)\left(S(z) + F_{\mathcal{A}}(z) + F_{\mathcal{B}}(z)\right)(z - \varepsilon_j)^2 = \frac{P_1(z)}{R(z)(z - \varepsilon_j)^{m_j - 2}}, \tag{8}$$

where $R(z)$ is a polynomial and $R(\varepsilon_j) \ne 0$ and $P_1(\varepsilon_j) \ne 0$. Then

$$\left| \frac{P_1(r\varepsilon_j)}{R(r\varepsilon_j)(r\varepsilon_j - \varepsilon_j)^{m_j - 2}} \right| \to \infty,$$

as $r \to 1^-$. For $z = r\varepsilon_j$ we have $|z - \varepsilon_j|^2 = |r\varepsilon_j - \varepsilon_j|^2 = (1 - r)^2$ and $S(z) = \sum_{n=0}^{\infty} \chi_S(n) z^n$, where $\chi_S(n)$ is the characteristic function of the set S (i.e. $\chi_S(n) = 1$, if $n \in S$ and $\chi_S = 0$, if $n \notin S$) we get the following estimation to the left hand side of (8) for $r < 1$

$$\begin{aligned}|3S(r\varepsilon_j)| \cdot |(S(r\varepsilon_j) + F_{\mathcal{A}}(r\varepsilon_j) + F_{\mathcal{B}}(r\varepsilon_j))| \cdot |r\varepsilon_j - \varepsilon_j|^2 \le \\ \le 3\left(\sum_{n=0}^{\infty} \chi(n)|r|^n\right)\left(\sum_{n=0}^{\infty} \chi(n)|r|^n + C_1\right) \cdot (1 - r)^2 < \\ < \frac{C_2}{(1 - r)^2} \cdot (1 - r)^2 = C_2,\end{aligned}$$

which is absurd.
Thus for some positive integer M we have $P_2(z) | (1 - z^M)^2$, so there is a polynomial $P_3(z)$ such that

$$3S(z)\left(S(z) + F_{\mathcal{A}}(z) + F_{\mathcal{B}}(z)\right) = \frac{P_3(z)}{(1 - z^M)^2}. \tag{9}$$

Multiplying equation (9) by 12 and adding $9\left(F_{\mathcal{A}}(z) + F_{\mathcal{B}}(z)\right)^2$ to it, we have

$$\left(6S(z) + 3F_{\mathcal{A}}(z) + 3F_{\mathcal{B}}(z)\right)^2 = \frac{P_4(z)}{(1 - z^M)^2}.$$

So

$$\left(6S(z) + 3F_{\mathcal{A}}(z) + 3F_{\mathcal{B}}(z)\right)^2 (1 - z^M)^2 = P_4(z).$$

We prove that $P_4(z) = (u(z))^2$, where $u(z)$ is a polynomial with integer coefficients.
Let

$$\left|(6S(z) + F_{\mathcal{A}}(z) + F_{\mathcal{B}}(z))^2\right| \cdot |(1 - z^M)^2| = \left|\sum_{n=0}^{\infty} g_n z^n\right|^2 = |P_4(z)|, \tag{10}$$

where $g_n \in \mathbb{Z}$. Since $P_4(z)$ is a polynomial, the integral $\int_0^{2\pi} |P_4(z)| \mathrm{d}\theta$ is bounded for $r \le 1$. On the other hand if there exist infinitely many n such that $g_n \neq 0$, that is $g_n^2 \ge 1$, then using the Parseval-formula we get

$$\int_0^{2\pi} \left| \sum_{n=0}^{\infty} g_n z^n \right|^2 \mathrm{d}\theta = \sum_{n=0}^{\infty} g_n^2 r^{2n} \to \infty,$$

as $r \to 1^-$, which is absurd. Thus the series $\sum_{n=0}^{\infty} g_n z^n = u(z)$ is a polynomial.

This means that there is an integer K such that if $n \ge K$, then $g_n = 0$, and according to formula (10) if $n \ge N_3$ then $g_n = 6\left(\chi(n) - \chi(n+M)\right) = 0$. So χ is periodic in M. Therefore there exist positive integer n_0, finite sets $F_{\mathcal{A}}$, $F_{\mathcal{B}}$, T with $F_{\mathcal{A}} \cup F_{\mathcal{B}} \subset \{0, 1, \dots, Mn_0 - 1\}$, $T \subset \{0, 1, \dots, M-1\}$ such that

$$A = F_{\mathcal{A}} \cup \{kM + t : k \ge n_0, t \in T\},$$

and

$$B = F_{\mathcal{B}} \cup \{kM + t : k \ge n_0, t \in T\}.$$

Hence the generating function of $\mathcal{A}$ and $\mathcal{B}$

$$A(z) = F_{\mathcal{A}}(z) + \frac{T(z) z^{n_0 M}}{1 - z^M},$$

and

$$B(z) = F_{\mathcal{B}}(z) + \frac{T(z) z^{n_0 M}}{1 - z^M}.$$

Then from (6) we have

$$A^3(z) - B^3(z) = \left(\frac{T(z) z^{n_0 M}}{1 - z^M} + F_{\mathcal{A}}(z) \right)^3 - \left(\frac{T(z) z^{n_0 M}}{1 - z^M} + F_{\mathcal{B}}(z) \right)^3 = Q(z). \qquad (11)$$

Thus

$$\frac{3T(z) z^{n_0 M}}{1 - z^M} \left(\frac{T(z) z^{n_0 M}}{1 - z^M} + F_{\mathcal{A}}(z) + F_{\mathcal{B}}(z) \right) \left(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z) \right) = P(z), \qquad (12)$$

that is

$$\frac{T(z) z^{n_0 M} \left(T(z) z^{n_0 M} + \left(F_{\mathcal{A}}(z) + F_{\mathcal{B}}(z) \right) (1 - z^M) \right) \left(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z) \right)}{(1 - z^M)^2} = R(z), \qquad (13)$$

where $R(z)$ is also a polynomial. Using $(1 - z^M, z^{n_0 M}) = 1$ we obtain that

$$(1 - z^M)^2 \left| T(z) \left(T(z) z^{n_0 M} + \left(F_{\mathcal{A}}(z) + F_{\mathcal{B}}(z) \right) (1 - z^M) \right) \left(F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z) \right) \right. \qquad (14)$$

that is

$$(1 - z^M)^2 | z^{n_0 M} (F_{\mathcal{A}}(z) - F_B(z)) T(z)^2 + (1 - z^M)(F_{\mathcal{A}}(z) + F_B(z))(F_{\mathcal{A}}(z) - F_B(z)) T(z). \qquad (15)$$

We prove that $1 - z^M | (F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)) T(z)$. By contradiction, assume that

$$1 - z^M \not| (F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z)) T(z).$$

This means that there exists an integer k, such that $k|M$ and

$$\Phi_k(z) \nmid (F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z))T(z)$$

(the polynomial $\Phi_k(z)$ denotes the kth cyclotomic polynomial). Then by (14) we get

$$\Phi_k(z)|T(z)z^{n_0 M} + (F_{\mathcal{A}}(z) + F_{\mathcal{B}}(z))\,(1 - z^M).$$

Thus $\Phi_k(z)|T(z)z^{n_0 M}$, but using that $(\Phi_k(z), z^{n_0 M}) = 1$ we get $\Phi_k(z)|T(z)$, which is absurd. Then

$$(1 - z^M)^2 \left|(1 - z^M)(F_{\mathcal{A}}(z) + F_B(z))(F_{\mathcal{A}}(z) - F_B(z))T(z)\right. ,$$

therefore by (15) we get that

$$(1 - z^M)^2 \left|z^{n_0 M}(F_{\mathcal{A}}(z) - F_B(z))T(z)^2\right. .$$

But using the fact that $\big((1 - z^M)^2, z^{n_0 M}\big) = 1$ this means that (3) holds, as desired. The other direction is the corollary of Theorem 1.■

4 Proof of Theorem 3.

Let $\mathcal{A}$ be a sparse, set which means that $\alpha(N) < N^{\frac{1}{p}}$ (here $\alpha(N) = |[0, N] \cap \mathcal{A}|$). Let $\mathcal{A} = \{a_1, a_2, \dots\}$. We prove by contradiction. Assume that $\mathcal{A}$, $\mathcal{B}$ are different sets and $R^{(1)}_{p,\mathcal{A}}(n) = R^{(1)}_{p,\mathcal{B}}(n)$ from a certain point on. Since $\alpha(a_k) = k < a_k^{1/p}$ it follows that $a_k > k^p$. The generating function of $\mathcal{A}$ is

$$\begin{aligned} A(r) = \sum_{a \in \mathcal{A}} r^a = \sum_{n=0}^{\infty} \chi_{\mathcal{A}}(n) r^n = \sum_{n=0}^{\infty} (\alpha(n) - \alpha(n-1))\, r^n = \\ = \sum_{n=1}^{\infty} \alpha(n) \left(r^n - r^{n+1}\right) = (1 - r) \sum_{n=0}^{\infty} \alpha(n) r^n = \\ = O\left((1 - r) \cdot (1 - r)^{-\frac{1}{p} - 1}\right) = O\left((1 - r)^{-\frac{1}{p}}\right), \quad (16) \end{aligned}$$

as $r \to 1^-$, where $\chi_{\mathcal{A}}(n)$ is the characteristic function of the set $\mathcal{A}$ (i.e. $\chi_{\mathcal{A}}(n) = 1$, if $n \in \mathcal{A}$ and $\chi_{\mathcal{A}}(n) = 0$, if $n \notin \mathcal{A}$).
Since $R^{(1)}_{p,\mathcal{A}}(n) = R^{(1)}_{p,\mathcal{B}}(n)$ it is clear that there is a polynomial $P(r)$ such that

$$A^p(r) - B^p(r) = P(r).$$

It is easy to see that there exists a positive integer N_0 such that $\mathcal{A} \cap [N_0, +\infty) = \mathcal{B} \cap [N_0, +\infty)$, because $R^{(1)}_{p,\mathcal{A}}(n) \equiv 0 \pmod p$ if $\frac{n}{p} \notin \mathcal{A}$, and $R^{(1)}_{p,\mathcal{A}}(n) \equiv 1 \pmod p$ if $\frac{n}{p} \in \mathcal{A}$. Similarly $R^{(1)}_{p,\mathcal{B}}(n) \equiv 0 \pmod p$ if $\frac{n}{p} \notin \mathcal{B}$, and $R^{(1)}_{p,\mathcal{B}}(n) \equiv 1 \pmod p$ if $\frac{n}{p} \in \mathcal{B}$. Thus $A(r)$ differs from $B(r)$ in a polynomial which means that

$$B(r) = O\left((1 - r)^{-\frac{1}{p}}\right), \quad (17)$$

as $r \to 1^-$, as well. So

$$(A(r) - B(r))\left(A^{p-1}(r) + \cdots + B^{p-1}(r)\right) = P(r). \tag{18}$$

Therefore there exist relatively prime polynomials $R(r)$ and $S(r)$ such that

$$R(r)\left(A^{p-1}(r) + \cdots + B^{p-1}(r)\right) = S(r). \tag{19}$$

As $r \to 1^-$ in (18) we get that $S(r)$ and $R(r)$ are bounded, and

$$A^{p-1}(r) + \cdots + B^{p-1}(r) \to \infty.$$

Therefore $r = 1$ must be the root of $R(r)$. Thus

$$R(r) = (1-r)Q(r).$$

Now we can write (19) into the following form

$$(1-r)Q(r)\left(A^{p-1}(r) + \cdots + B^{p-1}(r)\right) = S(r), \tag{20}$$

Since $Q(r)$ is a polynomial it is bounded. It follows from (16) and (17) that

$$A^{p-1}(r) + \cdots + B^{p-1}(r) = O\left((1-r)^{-\frac{p-1}{p}}\right).$$

So the order of the left hand side of (20) is $O\left((1-r)^{\frac{1}{p}}\right)$, as $r \to 1^-$. This means $S(r)$ tends to zero as $r \to 1^-$. So $S(r) = (1-r)T(r)$, and this contradicts to $(R(r), S(r)) = 1$. ■

5 Proof of Theorem 4.

The construction of these sets $\mathcal{A}$ and $\mathcal{B}$ are the following. Let n be a positive integer. Take the binary representation of n

$$n = \sum_{i=0}^{\lfloor \log_2(n) \rfloor} \beta_i 2^i,$$

where $\beta_i = 0$ or 1. Denote by $\text{Bin}(n) = \sum_{i=0}^{\lfloor \log_2(n) \rfloor} \beta_i$ the number of the ones in the binary representation of n. Let

$$F_{\mathcal{A}} := \{kH! | 0 \le k < 2^H, \text{Bin}(kH!) \equiv 0 \pmod 2\},$$

and

$$F_{\mathcal{B}} := \{kH! | 0 \le k < 2^H, \text{Bin}(kH!) \equiv 1 \pmod 2\}.$$

We will show that the sets

$$A = F_{\mathcal{A}} \cup \{H!2^H, H!2^H + 1, \dots\}$$

and

$$B = F_{\mathcal{B}} \cup \{H!2^H, H!2^H + 1, \dots\}$$

are suitable. Let h be a fixed integer, $2 \le h \le H$. Then we have

$$F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z) = \prod_{i=0}^{H-1} \left(1 - z^{H!2^i}\right),$$

and therefore

$$(1 - z^{h!}) \dots (1 - z^{2^{h-1}h!}) | F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z). \tag{21}$$

Hence

$$(1 - z) \dots (1 - z^{h-1})(1 - z^h) | F_{\mathcal{A}}(z) - F_{\mathcal{B}}(z).$$

The generating function of $R_{h,\mathcal{A}}^{(l)}(n)$, $l = 1, 2, 3$ can be written by sieve formula with suitable real numbers $C_{k_1,\dots,k_h}$:

$$\sum_{n=0}^{\infty} R_{h,\mathcal{A}}^{(l)}(n) z^n = \sum_{\substack{(k_1,\dots,k_h) \\ k_1+2k_2+\dots+hk_h=h \\ k_i \ge 0, i=1,\dots,h}} C_{k_1,\dots,k_h} \prod_{i=1}^{h} A(z^i)^{k_i}. \tag{22}$$

We would like to prove that there is a polynomial $P(z)$ such that

$$\sum_{n=0}^{\infty} R_{h,\mathcal{A}}^{(l)}(n) z^n - \sum_{n=0}^{\infty} R_{h,\mathcal{B}}^{(l)}(n) z^n = P(z). \tag{23}$$

From (22) we get that the left hand side of (23) is equivalent to the following

$$\sum_{\substack{(k_1,\dots,k_h) \\ k_1+2k_2+\dots+hk_h=h \\ k_i \ge 0, i=1,\dots,h}} C_{k_1,\dots,k_h} \left(\prod_{i=1}^{h} A(z^i)^{k_i} - \prod_{i=1}^{h} B(z^i)^{k_i} \right). \tag{24}$$

In view of

$$A(z) = F_{\mathcal{A}}(z) + \frac{z^{H!2^H}}{1 - z}$$

and

$$B(z) = F_{\mathcal{B}}(z) + \frac{z^{H!2^H}}{1 - z}$$

we get that (24) is equivalent to the following

$$\sum_{\substack{(k_1,\dots,k_h) \\ k_1+2k_2+\dots+hk_h=h \\ k_i \ge 0, i=1,\dots,h}} C_{k_1,\dots,k_h} \left(\prod_{i=1}^{h} \left(F_{\mathcal{A}}(z^i) + \frac{z^{iH!2^H}}{1 - z^i} \right)^{k_i} - \prod_{i=1}^{h} \left(F_{\mathcal{B}}(z^i) + \frac{z^{iH!2^H}}{1 - z^i} \right)^{k_i} \right). \tag{25}$$

It is enough to show that the difference of the products in (25) is polynomial for every h-tuple $(k_1,\dots,k_h)$. Let the h-tuple $(k_1,\dots,k_h)$ be fixed. Using the binomial theorem we get for suitable constants $D_{j_1,\dots,j_h}$ this expression is equal to the following

$$\left(\prod_{i=1}^{h}\sum_{j_i=0}^{k_i}\binom{k_i}{j_i}\left(F_{\mathcal{A}}(z^i)\right)^{j_i}\left(\frac{z^{iH!2^H}}{1-z^i}\right)^{k_i-j_i}\right)-\left(\prod_{i=1}^{h}\sum_{j_i=0}^{k_i}\binom{k_i}{j_i}\left(F_{\mathcal{B}}(z^i)\right)^{j_i}\left(\frac{z^{iH!2^H}}{1-z^i}\right)^{k_i-j_i}\right)=$$

$$=\sum_{\substack{(j_1,\dots,j_h)\\0\le j_i\le k_i,i=1,\dots,h}} D_{j_1,\dots,j_h}\left(\prod_{i=1}^{h}\left(\frac{z^{iH!2^H}}{1-z^i}\right)^{k_i-j_i}\right)\left(\prod_{i=1}^{h}\left(F_{\mathcal{A}}(z^i)\right)^{j_i}-\prod_{i=1}^{h}\left(F_{\mathcal{B}}(z^i)\right)^{j_i}\right). \quad (26)$$

We will show that

$$\left(\prod_{i=1}^{h}\left(\frac{z^{iH!2^H}}{1-z^i}\right)^{k_i-j_i}\right)\left(\prod_{i=1}^{h}\left(F_{\mathcal{A}}(z^i)\right)^{j_i}-\prod_{i=1}^{h}\left(F_{\mathcal{B}}(z^i)\right)^{j_i}\right)$$

is a polynomial. To show this we will prove that there is a polynomial $Q(z)$ such that

$$\prod_{i=1}^{h}\left(\frac{z^{iH!2^H}}{1-z^i}\right)^{k_i-j_i}=\frac{Q(z)}{(1-z)\dots(1-z^{h-1})(1-z^h)}, \quad (27)$$

and

$$(1-z)\dots(1-z^{h-1})(1-z^h)\,\Bigg|\,\prod_{i=1}^{h}\left(F_{\mathcal{A}}(z^i)\right)^{j_i}-\prod_{i=1}^{h}\left(F_{\mathcal{B}}(z^i)\right)^{j_i}. \quad (28)$$

To prove equation (27) it is enough to show that

$$\prod_{i=1}^{h}(1-z^i)^{k_i-j_i}\,\big|(1-z)\dots(1-z^{h-1})(1-z^h).$$

A root of the product $\prod_{i=1}^{h}(1-z^i)^{k_i-j_i}$ is a primitive ith roots of unity, for some $i\le h$. Let ε_i denote a primitive ith root of unity. The multiplicity of ε_i in polynomial $(1-z)\dots(1-z^{h-1})(1-z^h)$ is $\lfloor\frac{h}{i}\rfloor$. The multiplicity of ε_i in polynomial $\prod_{i=1}^{h}(1-z^i)^{k_i-j_i}$ is

$$(k_i-j_i)+(k_{2i}-j_{2i})+\dots\le k_i+k_{2i}+\dots.$$

We know that $k_1+2k_2+\dots+hk_h=h$, therefore

$$ik_i+ik_{2i}+\dots\le ik_i+2ik_{2i}+\dots\le 1k_1+2k_2+\dots hk_h=h,$$

that means

$$k_i+k_{2i}+\dots\le\left\lfloor\frac{h}{i}\right\rfloor,$$

which proves equation (27).
It remains to prove the following lemma, which verifies equation (28).

Lemma 3. *If* $(1-z)\dots(1-z^{h-1})(1-z^h)|F_{\mathcal{A}}(z)-F_{\mathcal{B}}(z)$ *then for all t-tuple* $(l_1,\dots,l_t)$

$$(1-z)\dots(1-z^{h-1})(1-z^h)\left|\prod_{i=1}^{t}\left(F_{\mathcal{A}}(z^i)\right)^{l_i}-\prod_{i=1}^{t}\left(F_{\mathcal{B}}(z^i)\right)^{l_i}\right. .$$

Proof. We prove by induction on t. If $t=1$ then we show that

$$(1-z)\dots(1-z^{h-1})(1-z^h)\left|(F_{\mathcal{A}}(z))^{l_1}-(F_{\mathcal{B}}(z))^{l_1}\right. . \tag{29}$$

Since

$$(F_{\mathcal{A}}(z))^{l_1}-(F_{\mathcal{B}}(z))^{l_1}=(F_{\mathcal{A}}(z)-F_{\mathcal{B}}(z))\left((F_{\mathcal{A}}(z))^{l_1-1}+\dots+(F_{\mathcal{B}}(z))^{l_1-1}\right),$$

we get that the case $t=1$ holds.
Now assume that the lemma holds for all t or less. For $t+1$ we need to show that

$$(1-z)\dots(1-z^{h-1})(1-z^h)\left|\prod_{i=1}^{t+1}\left(F_{\mathcal{A}}(z^i)\right)^{l_i}-\prod_{i=1}^{t+1}\left(F_{\mathcal{B}}(z^i)\right)^{l_i}\right. . \tag{30}$$

The right hand side of (30) is equal to

$$\begin{aligned}
&(F_{\mathcal{A}}(z))^{l_1}\dots\left(F_{\mathcal{A}}(z^{t+1})^{l_{t+1}}\right)-(F_{\mathcal{A}}(z))^{l_1}\dots\left(F_{\mathcal{A}}(z^t)\right)^{l_t}\left(F_{\mathcal{B}}(z^{t+1})\right)^{l_{t+1}}+\\
&\quad+(F_{\mathcal{A}}(z))^{l_1}\dots\left(F_{\mathcal{A}}(z^t)\right)^{l_t}\left(F_{\mathcal{B}}(z^{t+1})\right)^{l_{t+1}}-(F_{\mathcal{B}}(z))^{l_1}\dots\left(F_{\mathcal{B}}(z^{t+1})\right)^{l_{t+1}}=\\
&\qquad=(F_{\mathcal{A}}(z))^{l_1}\dots\left(F_{\mathcal{A}}(z^t)\right)^{l_t}\left(\left(F_{\mathcal{A}}(z^{t+1})\right)^{l_{t+1}}-\left(F_{\mathcal{B}}(z^{t+1})\right)^{l_{t+1}}\right)-\\
&\qquad\quad-\left(F_{\mathcal{B}}(z^{t+1})\right)^{l_{t+1}}\left((F_{\mathcal{A}}(z))^{l_1}\dots\left(F_{\mathcal{A}}(z^t)^{l_t}\right)-(F_{\mathcal{B}}(z))^{l_1}\dots\left(F_{\mathcal{B}}(z^t)\right)^{l_t}\right).
\end{aligned}$$

Because of our assumption the second term is divisible by $(1-z)\dots(1-z^{h-1})(1-z^h)$.
Since

$$(1-z)\dots(1-z^{h-1})(1-z^h)\left|(1-z^{t+1})\dots(1-z^{h(t+1)})\right.$$

and

$$(1-z^{t+1})\dots(1-z^{h(t+1)})\left|\left(F_{\mathcal{A}}(z^{t+1})\right)^{l_{t+1}}-\left(F_{\mathcal{B}}(z^{t+1})\right)^{l_{t+1}}\right),$$

which completes the induction. ■

6 Proof of Theorem 5.

We prove by contradiction. Assume that for infinite sets of nonnegative integers $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A}\neq\mathcal{B}$ there is an infinite sequence of integers $2\le h_1<h_2<\dots h_i<\dots$ and polynomials $P_i(r)$ such that

$$A^{h_i}(r)-B^{h_i}(r)=\sum_{n=0}^{\infty}\left(R^{(1)}_{h_i,\mathcal{A}}(n)-R^{(1)}_{h_i,\mathcal{B}}(n)\right)r^n=P_i(r).$$

Then

$$P_i(r)=A^{h_i}(r)-B^{h_i}(r)=(A(r)-B(r))\left(A^{h_i-1}(r)+A^{h_i-2}(r)B(r)+\dots+B^{h_i-1}(r)\right).$$

As $r \to 1^-$ we get

$$\frac{P_{i+1}(r)}{P_i(r)} = \frac{A^{h_i-1}(r) + A^{h_i-2}(r)B(r) + \dots + B^{h_i-1}(r)}{A^{h_{i+1}-1}(r) + A^{h_{i+1}-2}(r)B(r) + \dots + B^{h_{i+1}-1}(r)} \leq$$

$$\leq \frac{h_i \cdot \max\left\{A^{h_i-1}(r), B^{h_i-1}(r)\right\}}{\max\left\{A^{h_{i+1}-1}(r), B^{h_{i+1}-1}(r)\right\}} \to 0.$$

Let $P_i(r) = (1-r)^{m_i} Q_i(r)$, where m_i is a nonnegative integer and $Q_i(r)$ is a polynomial and $Q_i(1) \neq 0$. Thus

$$\frac{P_{i+1}(r)}{P_i(r)} = \frac{(1-r)^{m_{i+1}} Q_{i+1}(r)}{(1-r)^{m_i} Q_i(r)},$$

and $m_{i+1} < m_i$. We get that $m_1 > m_2 > \dots$, which is absurd. ■

References

[1] Yong-Gao Chen, Min Tang, Partitions of natural numbers with the same representation functions, *Journal of Number Theory*, Vol. 129 (2009), 2689-2695.

[2] Yong-Gao Chen, Bin Wang, On additive properties of two special sequences, *Acta Arithmetica*, Vol. 110.3 (2003), 299-303.

[3] G. Dombi, Additive properties of certain sets, *Acta Arithmetica*, Vol. 103.2 (2002), 137-146.

[4] L. Kronecker, Zwei sätse über Gleichungen mit ganzzahligen Coefficienten, *J. Reine Angew. Math.*, Vol. 53 (1857), 173-175.

[5] V. F. Lev, Reconstructing integer sets from their representation functions, *Electronic Journal of Combinatorics*, Vol. 11 (2004), R78.

[6] M. B. Nathanson, Representation functions of sequences in additive number theory, *Proceedings of the American Mathematical Society*, Vol. 72, (1978), 16-20.

[7] Cs. Sándor, Partitions of natural numbers and their representation functions, *Integers*, Vol. 4 (2004), A18.

[8] Min Tang, Partitions of the set of natural numbers and their representation functions, *Discrete Mathematics*, Vol. 308 (2008), 2614-2616.

[9] Quan-Hui Yang, Another Proof of Nathanson's Theorems, *Journal of Integer Sequences*, Vol. 14 (2011).