Groups, partitions and representation functions

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Abstract

Let G be a finite (abelian) group. In this paper we determine all subsets $A \subset G$ such that the number of solutions of $g = x + y$, $x, y \in A$ equals to the number of solutions of $g = x + y$, $x, y \in G \setminus A$. We discuss some related problems.

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1 Introduction

Let X be a semigroup, written additively. Let $A_1, \dots, A_h$ be subsets of X and let x be an element of X. We define the ordered representation function

$$R_{A_1+\dots+A_h}(x) = \#\{(a_1, \dots, a_h) \in A_1 \times \dots \times A_h : a_1 + \dots + a_h = x\}.$$

If $A_i = A$ for $i = 1, \dots, h$, then we write

$$R^{(1)}_{A,h}(x) = \#\{(a_1, \dots, a_h) : a_i \in A : a_1 + \dots + a_h = x\}.$$

Let X be an abelian semigroup, written additively. For $A \subset X$, let A^h denote the set of all h-tuples of A. Two h-tuples $(a_1, \dots, a_h) \in A^h$ and $(a'_1, \dots, a'_h) \in A^h$ are equivalent if there

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is a permutation $\alpha : \{1, \ldots, h\} \to \{1, \ldots, h\}$ such that $a_{\alpha(i)} = a'_i$ for $i = 1, \ldots, h$. Two other representation functions arise often and naturally in additive number theory. The unordered representation function $R^{(2)}_{A,h}(x)$ counts the number of equivalence classes of h-tuples $(a_1, \ldots, a_h)$ such that $a_1 + \cdots + a_h = x$. The unordered restricted representation function $R^{(3)}_{A,h}(x)$ counts the number of equivalence classes of h-tuples $(a_1, \ldots, a_h)$ of pairwise distinct elements of A such that $a_1 + \cdots + a_h = x$.

Alternative definitions for $R^{(2)}_{A,2}(x)$ and $R^{(3)}_{A,2}(x)$ are the following. Denote by

$$D_A(x) = \#\{a : a \in A, a + a = x\}$$

then

$$R^{(2)}_{A,2}(x) = \frac{1}{2}R^{(1)}_{A,2}(x) + \frac{1}{2}D_A(x)$$

and

$$R^{(3)}_{A,2}(x) = \frac{1}{2}R^{(1)}_{A,2}(x) - \frac{1}{2}D_A(x)$$

Let $\mathbb{N}$ be the set of nonnegative integers. Let $X = \mathbb{N}$. Answering a question of Sárközy, Lev [1] and independently Sándor [2] characterized all subsets $A \subset \mathbb{N}$ such that $R^{(2)}_{A,2}(n) = R^{(2)}_{\mathbb{N}\setminus A,2}(n)$ or $R^{(3)}_{A,2}(n) = R^{(3)}_{\mathbb{N}\setminus A,2}(n)$ from a certain point on. The precise theorems are the following.

Theorem (Lev, Sándor, 2004)**.** *Let $X = \mathbb{N}$. Let N be a positive integer. The equality $R^{(2)}_{\mathcal{A}}(n) = R^{(2)}_{\mathbb{N}\setminus\mathcal{A}}(n)$ holds for $n \geq 2N - 1$ if and only if $|A \cap [0, 2N - 1]| = N$ and $2m \in A \Leftrightarrow m \in A$, $2m + 1 \in A \Leftrightarrow m \notin A$ for $m \geq N$.*

Theorem (Lev, Sándor, 2004)**.** *Let $X = \mathbb{N}$. Let N be a positive integer. The equality $R^{(3)}_{\mathcal{A}}(n) = R^{(3)}_{\mathbb{N}\setminus\mathcal{A}}(n)$ holds for $n \geq 2N - 1$ if and only if $|A \cap [0, 2N - 1]| = N$ and $2m \in A \Leftrightarrow m \notin A$, $2m + 1 \in A \Leftrightarrow m \in A$ for $m \geq N$.*

Similar statement can not be formulated for the representation function $R^{(1)}_{\mathcal{A}}(n)$ because $R^{(1)}_{\mathcal{A}}(n)$ is odd if and only if $\frac{n}{2} \in \mathcal{A}$, therefore either $R^{(1)}_{\mathcal{A}}(2m)$ or $R^{(1)}_{\mathbb{N}\setminus\mathcal{A}}(2m)$ is odd.

A nontrivial result is the following in this direction.

Theorem 1. *Let $X = \mathbb{N}$. The equality $R^{(1)}_{A+B}(n) = R^{(1)}_{\mathbb{N}\setminus A+\mathbb{N}\setminus B}(n)$ holds from a certain point on if and only if $|\mathbb{N} \setminus (A \cup B))| = |A \cap B| < \infty$*

The modular questions were solved by Chen and Yang [3].

Theorem (Chen, Yang, 2012)**.** *Let $X = \mathbb{Z}_m$. The equality $R^{(1)}_{A,2}(n) = R^{(1)}_{\mathbb{N}\setminus A,2}(n)$ holds for all $n \in \mathbb{Z}_m$ if and only if m is even and $|A| = m/2$.*

Theorem (Chen, Yang, 2012)**.** *Let $X = \mathbb{Z}_m$. For $i \in \{2, 3\}$, the equality $R_A^{(i)} = R_{\mathbb{Z}_m \setminus A}^{(i)}(n)$ holds for all $n \in \mathbb{Z}_m$ if and only if m is even and $t \in A \Leftrightarrow t + m/2 \notin A$ for $t = 0, 1, \ldots, m/2 - 1$.*

We extend the first theorem to arbitrary finite group G and the second theorem to finite abelian group.

Theorem 2. *Let $X = G$ be a finite group. The equality $R_{A+B}(g) = R_{G\setminus A+G\setminus B}(g)$ holds for all $g \in G$ if and only if $|A| + |B| = |G|$*

A direct consequence is the following.

Corollary 1. *Let $X = G$ be a finite group. The equality $R_A^{(1)}(g) = R_{G\setminus A,2}^{(1)}(g)$ holds for all $g \in G$ if and only if $|G|$ is even and $|A| = |G|/2$.*

Theorem 3. *Let $X = G$ be a finite abelian group. For $i \in \{2, 3\}$, the equality $R_A^{(i)}(g) = R_{G\setminus A,2}^{(i)}(g)$ holds for all $g \in G$ if and only if $D_A(g) = D_{G\setminus A}(g)$ for every $g \in G$.*

2 Proof

Proof of Theorem 1. Denote by $S(x)$ the generating function of a subset $S \subset \mathbb{N}$, i.e. $S(x) = \sum_{s\in S} x^s$. The generating function of $R_{A+B}(n)$ is $A(x)B(x)$, while the generating function of $R_{\mathbb{N}\setminus A+\mathbb{N}\setminus B}(n)$ is $(\frac{1}{1-x} - A(x))(\frac{1}{1-x} - B(x))$. Hence the condition $R_{A+B}(n) = R_{\mathbb{N}\setminus A+\mathbb{N}\setminus B}(n)$ holds from a certain point on is equivalent to

$$A(x)B(x) - (\frac{1}{1-x} - A(x))(\frac{1}{1-x} - B(x)) = p(x),$$

where $p(x)$ is a polynomial. This is equivalent to

$$A(x) + B(x) = \frac{1}{1-x} + p(x)(1-x). \tag{1}$$

Let $A(x) + B(x) = \sum_{n=0}^{\infty} c_n x^n$, where $c_n = 0, 1$ or 2. The equation (1) holds if and only if $c_n = 1$ except for finitely many integer n and the number of n for which $c_n = 0$ is equal to the number of n for which $c_n = 2$. This is equivalent to the condition $|\mathbb{N} \setminus (A \cup B))| = |A \cap B| < \infty$. ■

Proof of Theorem 2. For a given $S \subset G$ denote by χ_S its characteristic function, that is $\chi_S(g) = 1$ if $g \in S$ and $\chi_S(g) = 0$ if $g \notin S$ for every $g \in G$. Then $\chi_{G\setminus S} = 1 - \chi_S$.

Obviously, for every $g \in G$ we have

$$R_{A+B}(g) - R_{G\setminus A+G\setminus B}(g) = \sum_{c\in G}\chi_A(c)\chi_B(-c+g) - \sum_{c\in G}\chi_{G\setminus A}(c)\chi_{G\setminus B}(-c+g) =$$

$$\sum_{c\in G}\chi_A(c)\chi_B(-c+g) - \sum_{c\in G}(1-\chi_A(c))(1-\chi_B(-c+g)) = \sum_{c\in G}\chi_A(c) + \sum_{c\in G}\chi_B(-c+g) - \sum_{c\in G}1 =$$

$$|A| + |B| - |G|.$$

Hence if $R_{A+B}(g) = R_{G\setminus A+G\setminus B}(g)$ for every $g \in G$ then $|A| + |B| = |G|$ and if $|A| + |B| = |G|$ then $R_{A+B}(g) = R_{G\setminus A+G\setminus B}(g)$ for every $g \in G$. ■

Proof of Theorem 3. We only prove the case $i = 2$, because the proof of case $i = 3$ are very similar. Obviously, for every fixed $g \in G$ we have

$$|A| = \#\{(a, y) : a \in A, y \in G, a + y = g\} =$$

$$\#\{(a, y) : a \in A, y \in A, a + y = g\} + \#\{(a, y) : a \in A, y \in G \setminus A, a + y = g\} =$$

$$R_{A+A}(g) + R_{A+G\setminus A}(g) = 2R^{(2)}_{A,2}(g) - D_A(g) + R_{A+G\setminus A}(g),$$

thus

$$R^{(2)}_{A,2}(g) = \frac{1}{2}|A| + \frac{1}{2}D_A(g) - \frac{1}{2}R_{A+G\setminus A}(g).$$

Similarly,

$$|G \setminus A| = \#\{(x, b) : x \in G, b \in G \setminus A, x + b = g\} =$$

$$\#\{(x, b) : x \in G \setminus A, b \in G \setminus A, x + b = g\} + \#\{(x, b) : x \in A, b \in G \setminus A, x + b = g\} =$$

$$R_{G\setminus A+G\setminus A}(g) + R_{A+G\setminus A}(g) = 2R^{(2)}_{G\setminus A,2}(g) - D_{G\setminus A}(g) + R_{A+G\setminus A}(g),$$

thus

$$R^{(2)}_{G\setminus A,2}(g) = \frac{1}{2}|G \setminus A| + \frac{1}{2}D_{G\setminus A}(g) - \frac{1}{2}R_{A+G\setminus A}(g).$$

Hence for every $g \in G$ we have

$$R^{(2)}_{A,2}(g) - R^{(2)}_{G\setminus A,2}(g) = \frac{1}{2}|A| + \frac{1}{2}D_A(g) - (\frac{1}{2}|G \setminus A| + \frac{1}{2}D_{G\setminus A}(g)) =$$

$$|A| - \frac{1}{2}|G| + \frac{1}{2}(D_A(g) - D_{G\setminus A}(g)).$$

Suppose that $R_{A,2}^{(2)}(g) = R_{G\setminus A,2}^{(2)}(g)$ for every $g \in G$. Then we have

$$\binom{|A|+1}{2} = \sum_{g\in G} R_{A,2}^{(2)}(g) = \sum_{g\in G} R_{G\setminus A,2}^{(2)}(g) = \binom{|G\setminus A|+1}{2},$$

therefore $|A| = |G \setminus A|$, that is $|A| = |G| - |A|$. Hence we get that $D_A(g) = D_{G\setminus A}(g)$ for every $g \in G$.

Finally, suppose that $D_A(g) = D_{G\setminus A}(g)$ for every $g \in G$. Then we have

$$|A| = \sum_{g\in G} D_A(g) = \sum_{g\in G} D_{G\setminus A}(g) = |G \setminus A|,$$

therefore $|A| = |G| - |A|$, thus we get that $R_{A,2}^{(2)}(g) = R_{G\setminus A,2}^{(2)}(g)$ for every $g \in G$, which completes the proof. ■

References

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