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A family of self-similar sets with overlaps

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ABSTRACT

In this note we consider a family of self-similar iterated function system on the line with overlapping cylinders. We point out that there exists an uncountable family of parameters for which the Hausdorff-dimension of the attractor is smaller than one although the similarity dimension is bigger than one.

1. INTRODUCTION

Our research was motivated by R. Tijdeman's question which was posed in Budapest January 2003:

'Is there a nonempty interval I such that every number $S \in I$ can be written in the form $S = \sum_{n=1}^{\infty} a_1 a_2 \cdots a_n$, where $a_n \in \{1/3, 1/2\}$?'

In this note we are going to study some more general problems. Let $\bar{\lambda} = \{0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m < 1\}$ and

$$\Lambda^{\bar{\lambda}} = \left\{ x : x = \sum_{n=1}^{\infty} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_n},\ i_n \in \{1, 2, \ldots, m\} \right\}.$$

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Define the maps $S_i(x) = \lambda_i x + \lambda_i$ for $i = 1, 2, \ldots, m$. It easy to see that

$$\Lambda^{\bar{\lambda}} = \bigcup_{i=1}^{m} S_i\big(\Lambda^{\bar{\lambda}}\big),$$

where $S_i(\Lambda^{\bar{\lambda}}) = \{\lambda_i x + \lambda_i \mid x \in \Lambda^{\bar{\lambda}}\}$. A set that is a union of a number of smaller similar copies of itself is called self-similar set. Denote $I^{\bar{\lambda}} = [\lambda_1/(1-\lambda_1), \lambda_m/(1-\lambda_m)]$. Then we have

$$(S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_N})\big(I^{\bar{\lambda}}\big) = \lambda_{i_1} + \lambda_{i_1}\lambda_{i_2} + \cdots + \lambda_{i_1}\cdots\lambda_{i_N} + \lambda_{i_1}\cdots\lambda_{i_N} I^{\bar{\lambda}}.$$

First we deal with the cases where either the sum of the contractions is less then one or the cylinders $S_i(I^{\bar{\lambda}})$, $S_{i+1}(I^{\bar{\lambda}})$ intersect each other for $i = 1, 2, \ldots, m-1$.

Theorem 1 (Hutchinson [3]). (a) *Let us suppose that* $\lambda_1 + \lambda_2 + \cdots + \lambda_m < 1$, *then the Hausdorff dimension* $\dim_{\mathrm{H}} \Lambda^{\bar{\lambda}} \leqslant t < 1$, *where* $\lambda_1^t + \lambda_2^t + \cdots + \lambda_m^t = 1$, *thus its Lebesgue measure* $\mathrm{Leb}(\Lambda^{\bar{\lambda}}) = 0$. *If the cylinders* $S_i(I^{\bar{\lambda}})$, $S_{i+1}(I^{\bar{\lambda}})$ *are disjoint, i.e.,* $\lambda_i/(1-\lambda_m) < \lambda_{i+1}/(1-\lambda_1)$, *for* $i = 1, \ldots, m-1$ *then* $\dim_{\mathrm{H}} \Lambda^{\bar{\lambda}} = t$.

(b) *If* $\lambda_i/(1-\lambda_m) \geqslant \lambda_{i+1}/(1-\lambda_1)$ *for* $i = 1, 2, \ldots, m-1$, *then* $\Lambda^{\bar{\lambda}} = I^{\bar{\lambda}}$.

Using the above theorem we can completely describe the case $m = 2$:

Corollary 1. (a) *For* $\lambda_1 + \lambda_2 < 1$, *i.e.,* $\lambda_1/(1-\lambda_2) < \lambda_2/(1-\lambda_1)$ *we have* $\dim_{\mathrm{H}} \Lambda^{\bar{\lambda}} = t < 1$, *where* $\lambda_1^t + \lambda_2^t = 1$, *therefore* $\mathrm{Leb}(\Lambda^{\bar{\lambda}}) = 0$.

(b) *If* $\lambda_1 + \lambda_2 \geqslant 1$, *i.e.,* $\lambda_1/(1-\lambda_2) \geqslant \lambda_2/(1-\lambda_1)$, *then* $\Lambda^{\bar{\lambda}} = [\lambda_1/(1-\lambda_1), \lambda_2/(1-\lambda_2)]$.

For $\lambda_1 = 1/k^2, \lambda_2 = 1/(k^2-1), \ldots, \lambda_{k^2-k+1} = 1/k$ we get a result of Tijdeman and Yuan (see [8], the case $k = 2$ was handled in [2]).

Corollary 2. *Let* $k > 1$ *be an integer. Let* $S \in [1/(k^2-1), 1/(k-1)]$. *Then there exist* $a_n \in \{k, k+1, \ldots, k^2\}$ *such that* $S = \sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_n}$.

In particular, we have a negative answer to Tijdeman's question since $1/3 + 1/2 < 1$.

The case $m = 3$ seems to be much more complicated. By Theorem 1 the condition $\lambda_1 + \lambda_2 + \lambda_3 < 1$ implies $\dim_{\mathrm{H}} \Lambda^{\bar{\lambda}} < 1$. But in contrast to the case $m = 2$ we can construct positive numbers $0 < \lambda_1 < \lambda_2 < \lambda_3 < 1$, where $\lambda_1 + \lambda_2 + \lambda_3 > 1$ and $\dim_{\mathrm{H}} \Lambda^{\bar{\lambda}} < 1$.

Theorem 2. *There exist uncountable many* $0 < \lambda_1 < \lambda_2 < \lambda_3 < 1$ *such that* $\lambda_1 + \lambda_2 + \lambda_3 > 1$ *and* $\dim_{\mathrm{H}} \Lambda^{\bar{\lambda}} < 1$.

Here we point out that we can construct an uncountable exceptional set of $\Lambda^{\bar{\lambda}}$ and the Hausdorff dimension of the exceptions is at least two. So far we have not been able to answer the following question:

Is it true that for Lebesgue a.e. $0 < \lambda_1 < \lambda_2 < \lambda_3 < 1$, $\lambda_1 + \lambda_2 + \lambda_3 > 1$ we have $\mathrm{Leb}(\Lambda^{\bar{\lambda}}) > 0$?

Fix the real numbers $\bar{\lambda} = \{0 < \lambda_1 < \lambda_2 < \lambda_3 < 1,\ \lambda_1 + \lambda_2 + \lambda_3 > 1\}$. Consider the following random series:

$$X_{\bar{\lambda}} = \sum_{n=1}^{\infty} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_n}, \quad i_n \in \{1, 2, 3\},$$

where the i_ns are chosen independently, with uniform distribution. Let $\nu_{\bar{\lambda}}$ be the distribution of $X_{\bar{\lambda}}$. If we could prove that $\nu_{\bar{\lambda}}$ is absolutely continuous for a.e. $\bar{\lambda}$ then we could answer our question affirmatively.

Similar problems were introduced by Keane, Smorodinsky, Solomyak (see [4]):

Is the Hausdorff dimension $\dim_{\mathrm{H}}(\Lambda(\lambda))$ of the parameter family of Cantor sets

$$\Lambda(\lambda) = \left\{ \sum_{k=1}^{\infty} i_k \lambda^k \mid i_k \in \{0, 1, 3\} \right\}$$

continuous on the interval $\lambda \in [1/4, 1/3]$?

In this case $\Lambda(\lambda) = \bigcup_{i=0,1,3} S_i^{\lambda}(\Lambda(\lambda))$, where $S_i^{\lambda}(\Lambda(\lambda)) = \lambda x + i$. Here all three maps share the same contractions but the translations are different. In our case ($\Lambda^{\bar{\lambda}} = \bigcup_{i=1}^{m} S_i(\Lambda^{\bar{\lambda}})$) however the contractions and translations are different in all three maps. For the K-S-S problem, Pollicott and Simon (see [5]) proved that for almost all $\lambda \in [1/4, 1/3]$ (with respect to the Lebesgue-measure) we have $\dim_{\mathrm{H}}(\Lambda(\lambda)) = -\frac{\log 3}{\log \lambda}$.

A similar problem is the following question of P. Erdős: Let $\lambda \in [0, 1)$ and

$$Y_{\lambda} = \sum_{n=0}^{\infty} \pm \lambda^n$$

where the signs are chosen independently, the plus sign with probability $1/2$ and the minus sign with probability $1/2$. Let ν_{λ} be the distribution of Y_{λ}. This is called a Bernoulli convolution since ν_{λ} is the infinite convolution product of $(\delta_{-\lambda^n} + \delta_{\lambda^n})/2$. A question which has been intensively studied since the 1930s is Erdős' question: for which λ is the measure ν_{λ} absolutely continuous with respect to the Lebesgue measure. It is easy to see that for $F_{\lambda}(x) = \nu_{\lambda}(-\infty, x)$:

$$F_{\lambda}(x) = \frac{1}{2}\left[F_{\lambda}\left(\frac{x-1}{\lambda}\right) + F_{\lambda}\left(\frac{x+1}{\lambda}\right) \right].$$

That is ν_{λ} is the self-similar measure for the iterated function system $\{\lambda x - 1, \lambda x + 1\}$ with probability $(1/2, 1/2)$. B. Solomyak proved (see [7]) (using many ideas from the work of Pollicott and Simon) that ν_{λ} is absolutely continuous for a.e. $\lambda \in [1/2, 1)$ (the known exceptions are the so called Pisot numbers). This problem can be interpreted as

$$Y_{\lambda} = \sum_{n=1}^{\infty} \lambda_1 \lambda_2 \cdots \lambda_n,$$

where $\lambda_n = \lambda$ with probability $1/2$ and $\lambda_n = -\lambda$ with probability $1/2$. We do not know yet if the exceptional sets in the two examples above are uncountable. Some experts strongly believe that these exceptional sets are countable in fact [6].

The main result of this paper is to construct a family of fractals of overlapping where we can prove the existence of an uncountable set.

2. PROOFS

Theorem 1 is well known. For the convenience of the reader we present its proof except for the last statement of part (a), which is somewhat lengthy (see also [1]).

2.1. Proof of Theorem 1

(a) Fix the positive integer N. Then we have

$$\Lambda^{\bar{\lambda}} = \bigcup_{\substack{(i_1,i_2,\ldots,i_N)\\ i_j\in\{1,2,\ldots,m\}}} (S_{i_1}\circ S_{i_2}\circ\cdots\circ S_{i_N})\big(\Lambda^{\bar{\lambda}}\big).$$

Let $d = \lambda_m/(1-\lambda_m) - \lambda_1/(1-\lambda_1)$. Then $\operatorname{diam}(S_{i_1}\circ S_{i_2}\circ\cdots\circ S_{i_N})(\Lambda^{\bar{\lambda}}) = \lambda_{i_N}\lambda_{i_{N-1}}\cdots\lambda_{i_1}d$ and for any real number s we have

$$\begin{aligned}
&\sum_{\substack{(i_1,i_2,\ldots,i_N)\\ i_j\in\{1,2,\ldots,m\}}} \big(\operatorname{diam}(S_{i_1}\circ S_{i_2}\circ\cdots\circ S_{i_N})\big(\Lambda^{\bar{\lambda}}\big)\big)^s\\
&\quad= \sum_{\substack{(i_1,i_2,\ldots,i_N)\\ i_j\in\{1,2,\ldots,m\}}} \lambda_{i_N}^s\lambda_{i_{N-1}}^s\cdots\lambda_{i_1}^s d^s = \big(\lambda_1^s+\lambda_2^s+\cdots+\lambda_m^s\big)^N d^s,
\end{aligned}$$

which proves that the Hausdorff dimension $\dim_{\mathrm{H}}\Lambda^{\bar{\lambda}} \leqslant t < 1$ where $\lambda_1+\lambda_2+\lambda_3 > 1$ and $\dim_{\mathrm{H}}\Lambda^{\bar{\lambda}} < 1$.

(b) Using the condition $\lambda_i/(1-\lambda_m) \geqslant \lambda_{i+1}/(1-\lambda_1)$ for $i = 1, 2, \ldots, m-1$ we have $I^{\bar{\lambda}} = \bigcup_{i=1}^m S_i(I^{\bar{\lambda}})$, but the attractor of the system $\{S_i\}_{i=1}^m$ is unique, thus $\Lambda^{\bar{\lambda}} = I^{\bar{\lambda}} = [\lambda_1/(1-\lambda_1), \lambda_m/(1-\lambda_m)]$, which proves part (b).

2.2. Proof of Theorem 2

Choose $1/3 < \lambda_1 < \lambda_2 < 1/3 + \varepsilon/6$, with ε sufficiently small, and put $\lambda_3 = \lambda_2 + \lambda_2/\lambda_1 - 1$.

Fix the positive integer N. Let $d = \lambda_3/(1-\lambda_3) - \lambda_1/(1-\lambda_1)$. For $1 \leqslant i_j \leqslant 3$, $j = 1, 2, \ldots, N$ we have

$$(S_{i_1}\circ S_{i_2}\circ\cdots\circ S_{i_N})(x) = \sum_{n=1}^{N} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n} + \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_N}x$$

and

$$\operatorname{diam}(S_{i_1}\circ S_{i_2}\circ\cdots\circ S_{i_N})\big(I^{\bar{\lambda}}\big) = \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_N}d \leqslant \left(\frac{1}{3}+\varepsilon\right)^N d.$$

Obviously, we have

$$\Lambda^{\bar{\lambda}} \subset \bigcup_{\substack{(i_1,i_2,\dots,i_N)\\ i_j\in\{1,2,3\}}} (S_{i_1}\circ S_{i_2}\circ\cdots\circ S_{i_N})\big(I^{\bar{\lambda}}\big).$$

We will separate this union into two parts:

$$T_N^{(1)} = \bigcup_{\substack{(i_1,i_2,\dots,i_N)\\ i_j\in\{1,2,\dots,m\}}} (S_{i_1}\circ S_{i_2}\circ\cdots\circ S_{i_N})\big(I^{\bar{\lambda}}\big),$$

where in the vector $(i_1, i_2, \dots, i_N)$ there are either at most cN j's for which $(i_j, i_{j+1}, i_{j+2}) = (1, 3, 2)$ or at most cN j's for which $(i_j, i_{j+1}, i_{j+2}) = (2, 1, 3)$. If the positive number c is small enough then a standard argument (for example by the Markov inequality) shows that $T_N^{(1)}$ is a union of at most $(c_1 3)^N$ intervals, where $c_1 < 1$. Thus the set $T_N^{(1)}$ can be covered by intervals $I_1^{(1)}, I_2^{(1)}, \dots, I_K^{(1)}$, where $\operatorname{diam} I_j^{(1)} \leqslant (1/3+\varepsilon)^N d$ and $K \leqslant (c_1 3)^N$.

The second part is

$$T_N^{(2)} = \bigcup_{\substack{(i_1,i_2,\dots,i_N)\\ i_j\in\{1,2,3\}}} (S_{i_1}\circ S_{i_2}\circ\cdots\circ S_{i_N})\big(I^{\bar{\lambda}}\big),$$

where in the vector $(i_1, i_2, \dots, i_N)$ there are at least cN j's for which $(i_j, i_{j+1}, i_{j+2}) = (1, 3, 2)$ and at least cN j's for which $(i_j, i_{j+1}, i_{j+2}) = (2, 1, 3)$. We know that

$$(S_{i_1}\circ S_{i_2}\circ\cdots\circ S_{i_N})\big(I^{\bar{\lambda}}\big) = \sum_{n=1}^{N} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n} + \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_N} I^{\bar{\lambda}}.$$

It is easy to see that the condition $\lambda_1 + \lambda_1\lambda_3 = \lambda_2 + \lambda_2\lambda_1$ implies that in the vector $(i_1, i_2, \dots, i_N)$ replacing $(i_j, i_{j+1}, i_{j+2}) = (1, 3, 2)$ by $(i_j, i_{j+1}, i_{j+2}) = (2, 1, 3)$ we get the same interval

$$\sum_{n=1}^{N} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n} + \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_N} I^{\bar{\lambda}}.$$

Therefore there is a $c_2 > 0$ such that the interval

$$\left[\sum_{n=1}^{N} \lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_n}, \sum_{n=1}^{N} \lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_n} + \lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_N}\left(\frac{1}{3}+\varepsilon\right)^N\right]$$

contains at least $2^{c_2 N}$ intervals

$$(S_{i_1}\circ S_{i_2}\circ\cdots\circ S_{i_N})\big(I^{\bar{\lambda}}\big).$$

Thus the set $T_N^{(2)}$ can be covered by intervals $I_1^{(2)}, I_2^{(2)}, \dots, I_L^{(2)}$, where $\operatorname{diam} I_j^{(2)} \leqslant (1/3+\varepsilon)^N d$ and $L \leqslant 3^N/(2^{c_2 N})$.

So if the positive number c is small enough then there exist positive numbers $c_1 < 1, c_2$ such that for $s > 0$

$$\sum_{i=1}^{K} \bigl(\operatorname{diam} I_i^{(1)}\bigr)^s + \sum_{i=1}^{L} \bigl(\operatorname{diam} I_i^{(2)}\bigr)^s \leqslant (c_1 3)^N \left(\frac{1}{3}+\varepsilon\right)^{Ns} d^s + \frac{3^N}{2^{c_2 N}} \left(\frac{1}{3}+\varepsilon\right)^{Ns} d^s,$$

where c_1, c_2 do not depend on ε and s. For some $s < 1$ and small enough ε this expression tends to zero as $N \to \infty$, which completes the proof.

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