14. gyakorlat - Hármas integrál

2014. május 22.

1. $D = \{(x, y, z) : 0 \leq x \leq 1, 0 \leq y \leq 1 - x, 0 \leq z \leq 1 - x - y\}$. Ekkor

$$\iiint_D f(x,y,z)dxdydz = \int_{x=0}^{1}\left(\int_{y=0}^{1-x}\left(\int_{z=0}^{1-x-y} x+y+zdz\right)dy\right)dx$$

A belső integrál:

$$\int_{z=0}^{1-x-y} x+y+zdz = \left[xz + yz + \frac{z^2}{2}\right]_0^{z=1-x-y} = x(1-x-y)+y(1-x-y)+\frac{(1-x-y)^2}{2} = \frac{1}{2}-xy-x^2-y^2,$$

így

$$\int_{x=0}^{1}\left(\int_{y=0}^{1-x}\left(\int_{z=0}^{1-x-y} x+y+zdz\right)dy\right)dz = \int_{x=0}^{1}\left(\int_{y=0}^{1-x}\frac{1}{2}-xy-x^2-y^2dy\right)dx$$

Itt a belső integrál:

$$\int_{y=0}^{1-x}\frac{1}{2}-xy-x^2-y^2dy = \left[\frac{1}{2}y - x\frac{y^2}{2} - x^2y - \frac{y^3}{3}\right]_0^{y=1-x} =$$

$$\frac{1}{2}(1-x) - x\frac{(1-x)^2}{2} - x(1-x) - \frac{(1-x)^3}{3} = \frac{1}{2}+\frac{3}{2}x - x^2 + \frac{5}{6}x^3$$

Innen

$$\int_{x=0}^{1}\left(\int_{y=0}^{1-x}\frac{1}{2}-xy-x^2-y^2dy\right)dx = \int_{x=0}^{1}\frac{1}{2}+\frac{3}{2}x-x^2+\frac{5}{6}x^3dx = \left[\frac{1}{2}x+\frac{3}{4}x^2-\frac{x^3}{3}+\frac{5}{24}x^4\right]_0^{x=1} = \frac{9}{8}$$

2. $D = \{(x, y, z) : -1 \leq x \leq 1, x^2 \leq y \leq 1, -1 \leq z \leq 1\}$. Ekkor

$$\iiint_D f(x,y,z)dxdydz = \int_{x=-1}^{1}\left(\int_{y=x^2}^{1}\left(\int_{z=-1}^{1} ydz\right)dy\right)dx$$

A belső integrál 2y, mivel a függvény nem tartalmaz z-t, ezért az integrációs út hosszával kell szorozni a függvényt. Így

$$\int_{x=-1}^{1}\left(\int_{y=x^2}^{1}\left(\int_{z=-1}^{1} ydz\right)dy\right)dx = \int_{x=-1}^{1}\left(\int_{y=x^2}^{1} 2ydy\right)dx.$$

Itt a belső integrál:

$$\int_{y=x^2}^{1} 2ydy = \left[y^2\right]_{x^2}^{y=1} = 1 - x^4,$$

így

$$\int_{x=-1}^{1}\left(\int_{y=x^2}^{1} 2ydy\right)dx = \int_{x=-1}^{1} 1 - x^4dx = \left[x - \frac{x^5}{5}\right]_{-1}^{x=1} = \frac{8}{5}$$

3. (a) Hengerkoordinátákat alkalmazunk: $x = r\cos\varphi, y = r\sin\varphi, z$, ahol $0 \leq r \leq 2$, $0 \leq \varphi \leq 2\pi$, $0 \leq z \leq 1$. Ekkor

$$\iiint_D f(x,y,z)dxdydz = \int_{r=0}^{2}\left(\int_{\varphi=0}^{2\pi}\left(\int_{z=0}^{1} r^2\cos^2\varphi \cdot rdz\right)d\varphi\right)dr$$

A belső integrál kiszámításánál mivel a függvény nem tartalmaz z-t, ezért az integrációs út hosszával kell szorozni a függvényt. Így

$$\int_{z=0}^{1} r^2\cos^2\varphi \cdot rdz = r^3\cos^2\varphi$$

Így

$$\int_{r=0}^{2}\left(\int_{\varphi=0}^{2\pi}\left(\int_{z=0}^{1} r^2\cos^2\varphi \cdot rdz\right)d\varphi\right)dr = \int_{r=0}^{2}\left(\int_{\varphi=0}^{2\pi} r^3\cos^2\varphi d\varphi\right)dr = \int_{r=0}^{2}\left(\int_{\varphi=0}^{2\pi} r^3\frac{1+\cos 2\varphi}{2}d\varphi\right)dr.$$

Most a belső integrál

$$\int_{\varphi=0}^{2\pi} r^3\frac{1+\cos 2\varphi}{2}d\varphi = \left[\frac{1}{2}r^3\varphi + \frac{r^4}{4}\sin 2\varphi\right]_0^{\varphi=2\pi} = r^3\pi$$

. Ezért

$$\int_{r=0}^{2}\left(\int_{\varphi=0}^{2\pi} r^3\frac{1+\cos 2\varphi}{2}d\varphi\right)dr = \int_{r=0}^{2} r^3\pi dr = \left[\frac{r^4\pi}{4}\right]_0^{r=2} = 4\pi$$

(b) Hengerkoordinátákat alkalmazunk: $x = r\cos\varphi, y = r\sin\varphi, z$, ahol $0 \leq r \leq 2$, $0 \leq \varphi \leq 2\pi$, $0 \leq z \leq 1$. Ekkor

$$\iiint_D f(x,y,z)dxdydz = \int_{r=0}^{2}\left(\int_{\varphi=0}^{2\pi}\left(\int_{z=0}^{1} \frac{z}{1+r^2} \cdot rdz\right)d\varphi\right)dr$$

A belső integrál:

$$\int_{z=0}^{1} \frac{z}{1+r^2} \cdot rdz = \left[\frac{z^2}{2}\frac{r}{1+r^2}\right]_0^{z=1} = \frac{r}{2(1+r^2)}$$

Így

$$\int_{r=0}^{2}\left(\int_{\varphi=0}^{2\pi}\left(\int_{z=0}^{1} \frac{z}{1+r^2} \cdot rdz\right)d\varphi\right)dr = \int_{r=0}^{2}\left(\int_{\varphi=0}^{2\pi} \frac{r}{2(1+r^2)}d\varphi\right)dr$$

Most a belső integrál nem tartalmaz φ-t, emiatt ennek a kiszámításánal az integrációs út hosszával kell szoroznunk, ami azt adja, hogy

$$\int_{r=0}^{2}\left(\int_{\varphi=0}^{2\pi}\frac{r}{2(1+r^2)}d\varphi\right)dr=\int_{r=0}^{2}\frac{r\pi}{1+r^2}dr=\left[\frac{\pi}{2}\ln(1+r^2)\right]_0^{r=2}=\frac{\pi}{2}\ln 5$$

(c) Hengerkoordinátákat alkalmazunk: $x=r\cos\varphi, y=r\sin\varphi, z$, ahol $0\le r\le 2$, $0\le\varphi\le 2\pi$, $0\le z\le 1$. Ekkor

$$\iiint_D f(x,y,z)dxdydz=\int_{r=0}^{2}\left(\int_{\varphi=0}^{2\pi}\left(\int_{z=0}^{1}e^{r^2}\cdot rdz\right)d\varphi\right)dr=$$

$$\int_{r=0}^{2}\left(\int_{\varphi=0}^{2\pi}e^{r^2}\cdot rd\varphi\right)dr=\int_{r=0}^{2}2r\pi e^{r^2}dr=\left[\pi e^{r^2}\right]_0^{r=1}=\pi e^4-\pi$$

(d) Hengerkoordinátákat alkalmazunk: $x=r\cos\varphi, y=r\sin\varphi, z$, ahol $0\le r\le 2$, $0\le\varphi\le 2\pi$, $0\le z\le 1$. Ekkor

$$\iiint_D f(x,y,z)dxdydz=\int_{r=0}^{2}\left(\int_{\varphi=0}^{2\pi}\left(\int_{z=0}^{1}\sqrt{1+r^2}\cdot rdz\right)d\varphi\right)dr=\int_{r=0}^{2}\left(\int_{\varphi=0}^{2\pi}\sqrt{1+r^2}\cdot rd\varphi\right)dr=$$

$$\int_{r=0}^{2}2\pi r\sqrt{1+r^2}dr=\left[\pi\frac{(1+r^2)^{3/2}}{3/2}\right]_0^{r=2}=\frac{2\pi}{3}(5^{3/2}-1)$$

4. Hengerkoordinátákat alkalmazunk: $x=r\cos\varphi, y=r\sin\varphi, z$, ahol $0\le r\le 1$, $\frac{\pi}{4}\le\varphi\le\frac{5\pi}{4}$, $0\le z\le 1+r\cos\varphi$. Ekkor

$$\iiint_D f(x,y,z)dxdydz=\int_{r=0}^{1}\left(\int_{\varphi=\frac{\pi}{4}}^{\frac{5\pi}{4}}\left(\int_{z=0}^{1+r\cos\varphi}z\cdot rdz\right)d\varphi\right)dr$$

A belső integrál:

$$\int_{z=0}^{1+r\cos\varphi}z\cdot rdz=\left[\frac{z^2r}{2}\right]_0^{z=1+r\cos\varphi}=r\frac{(1+r\cos\varphi)^2}{2}=\frac{1}{2}r+r^2\cos\varphi+\frac{1}{2}r^3\cos^2\varphi=\frac{1}{2}r+r^2\cos\varphi+\frac{1}{2}r^3\frac{1+\cos 2\varphi}{2},$$

ezért

$$\int_{r=0}^{1}\left(\int_{\varphi=\frac{\pi}{4}}^{\frac{5\pi}{4}}\left(\int_{z=0}^{1+r\cos\varphi}z\cdot rdz\right)d\varphi\right)dr=\int_{r=0}^{1}\left(\int_{\varphi=\frac{\pi}{4}}^{\frac{5\pi}{4}}\frac{1}{2}r+r^2\cos\varphi+\frac{1}{4}r^3+\frac{1}{4}r^3\cos 2\varphi d\varphi\right)dr=$$

$$\int_{r=0}^{1}\left[\frac{1}{2}r\varphi+r^2\sin\varphi+\frac{1}{4}r^3\varphi+\frac{1}{8}r^3\sin 2\varphi\right]_{\frac{\pi}{4}}^{\varphi=\frac{5\pi}{4}}dr=\int_{r=0}^{1}\frac{r\pi}{2}-\sqrt{2}r^2+\frac{r^3\pi}{4}dr=$$

$$\left[\frac{r^2\pi}{4}-\sqrt{2}\frac{r^3}{3}+\frac{r^4\pi}{16}\right]_0^{r=1}=\frac{5\pi}{16}-\frac{\sqrt{2}}{3}.$$

5. Most $D = \{(x,y,z) : x^2+y^2 \leq 1, x^2+y^2 \leq z \leq 2-x^2-y^2\}$ és mivel térfogatot számolunk, ezért $f(x,y,z) = 1$. Hengerkoordinátákat alkalmazunk: $x = r\cos\varphi, y = r\sin\varphi, z$, ahol $0 \leq r \leq 1$, $0 \leq \varphi \leq 2\pi$, $r^2 \leq z \leq 2-r^2$. Ekkor

$$\iiint_D f(x,y,z)dxdydz = \int_{r=0}^{1}\left(\int_{\varphi=0}^{2\pi}\left(\int_{z=r^2}^{2-r^2} 1\cdot rdz\right)d\varphi\right)dr =$$

$$\int_{r=0}^{1}\left(\int_{\varphi=0}^{2\pi}(2-2r^2)\cdot rd\varphi\right)dr = \int_{r=0}^{1} 4r\pi - 4r^3\pi dr =$$

$$\left[2r^2\pi - r^4\pi\right]_0^{r=1} = \pi$$

6. Hengerkoordinátákat alkalmazunk: $x = r\cos\varphi, y = r\sin\varphi, z$, ahol $0 \leq r \leq 2$, $0 \leq \varphi \leq 2\pi$, $r^2 \leq z \leq 4$. Ekkor

$$\iiint_D f(x,y,z)dxdydz = \int_{r=0}^{2}\left(\int_{\varphi=0}^{2\pi}\left(\int_{z=r^2}^{4} z\cdot rdz\right)d\varphi\right)dr =$$

$$\int_{r=0}^{2}\left(\int_{\varphi=0}^{2\pi}\left[\frac{z^2r}{2}\right]_{r^2}^{4}d\varphi\right)dr = \int_{r=0}^{2}\left(\int_{\varphi=0}^{2\pi}8r-\frac{r^5}{2}d\varphi\right)dr = \int_{r=0}^{2}16r\pi - r^5\pi dr =$$

$$\left[8r^2\pi - \frac{r^6\pi}{6}\right]_0^{r=2} = \frac{64\pi}{3}$$

7. Gömbkoordinátákat alkalmazunk: $x = r\sin\alpha\cos\beta$, $y = r\sin\alpha\sin\beta$, $z = r\cos\alpha$, $0 \leq \alpha \leq \pi$, $0 \leq \beta \leq 2\pi$, $0 \leq r \leq 1$.

(a)

$$\iiint_D f(x,y,z)dxdydz = \int_{r=0}^{1}\left(\int_{\alpha=0}^{\pi}\left(\int_{\beta=0}^{2\pi} r\cdot r^2\sin\alpha d\beta\right)d\alpha\right)dr =$$

$$\int_{r=0}^{1}\left(\int_{\alpha=0}^{\pi}2\pi r^3\sin\alpha d\alpha\right)dr = \int_{r=0}^{1}\left[-2\pi r^3\cos\alpha\right]_0^{\alpha=\pi}dr =$$

$$\int_{r=0}^{1}4r^3\pi dr = \left[r^4\pi\right]_0^{r=1} = \pi$$

(b)

$$\iiint_D f(x,y,z)dxdydz = \int_{r=0}^{1}\left(\int_{\alpha=0}^{\pi}\left(\int_{\beta=0}^{2\pi} e^{r^3}\cdot r^2\sin\alpha d\beta\right)d\alpha\right)dr =$$

$$\int_{r=0}^{1}\left(\int_{\alpha=0}^{\pi}2\pi e^{r^3}r^2\sin\alpha d\alpha\right)dr = \int_{r=0}^{1}\left[-2\pi e^{r^3}r^2\cos\alpha\right]_0^{\alpha=\pi}dr =$$

$$\int_{r=0}^{1}4r^2\pi e^{r^3}dr = \left[\frac{4\pi e^{r^3}}{3}\right]_0^{r=1} = \frac{4\pi}{3}(e-1)$$

8. Gömbkoordinátákat alkalmazunk: $x = r\sin\alpha\cos\beta$, $y = r\sin\alpha\sin\beta$, $z = r\cos\alpha$, $0 \le \alpha \le \frac{\pi}{2}$, $0 \le \beta \le \frac{\pi}{2}$, $0 \le r \le 1$.

(a)

$$\iiint_D f(x,y,z)dxdydz = \int_{r=0}^{1}\left(\int_{\alpha=0}^{\frac{\pi}{2}}\left(\int_{\beta=0}^{\frac{\pi}{2}} r\sin\alpha\cos\beta\cdot r\sin\alpha\sin\beta\cdot r\cos\alpha\cdot r^2\sin\alpha d\beta\right)d\alpha\right)dr =$$

$$\int_{r=0}^{1}\left(\int_{\alpha=0}^{\frac{\pi}{2}}\left(\int_{\beta=0}^{\frac{\pi}{2}} r^5\sin^3\alpha\cos\alpha\sin\beta\cos\beta\, d\beta\right)d\alpha\right)dr =$$

$$\int_{r=0}^{1}\left(\int_{\alpha=0}^{\frac{\pi}{2}}\left[r^5\sin^3\alpha\cos\alpha\frac{\sin^2\beta}{2}\right]_0^{\beta=\frac{\pi}{2}}d\alpha\right)dr = \int_{r=0}^{1}\left(\int_{\alpha=0}^{\frac{\pi}{2}}\frac{1}{2}r^5\sin^3\alpha\cos\alpha\right)dr =$$

$$\int_{r=0}^{1}\left[\frac{1}{8}r^5\sin^4\alpha\right]_0^{\alpha=\frac{\pi}{2}}dr = \int_{r=0}^{1}\frac{1}{8}r^3dr =$$

$$\left[\frac{1}{32}r^4\right] = \frac{1}{32}$$

9. (a) Gömbkoordinátákat alkalmazunk: $x = r\sin\alpha\cos\beta$, $y = r\sin\alpha\sin\beta$, $z = r\cos\alpha$, $0 \le \alpha \le \pi$, $0 \le \beta \le 2\pi$, $0 \le r \le 1$. Ekkor a tömeg:

$$\iiint_D f(x,y,z)dxdydz = \int_{r=0}^{1}\left(\int_{\alpha=0}^{\pi}\left(\int_{\beta=0}^{2\pi} r\cdot r^2\sin\alpha d\beta\right)d\alpha\right)dr =$$

$$\int_{r=0}^{1}\left(\int_{\alpha=0}^{\pi}2\pi r^3\sin\alpha d\alpha\right)dr = \int_{r=0}^{1}\left[-2r^3\pi\cos\alpha\right]_0^{\alpha=\pi}dr =$$

$$\int_{r=0}^{1}4r^3\pi dr = \left[r^4\pi\right]_0^{r=1} = \pi$$

(b) A gömbből megmaradó rész térfogatát megkapjuk, ha a gömb térfogatából kivonjuk az átfúrt rész térfogatát:

$$V_{\text{megmaradó}} = V_{\text{gömb}} - V_{\text{átfúrt}}$$

Tudjuk, hogy az egység sugarú gömb térfogata: $V_{\text{gömb}} = \frac{4\pi}{3}$. Az átfúrt rész térfogatát hengerkoordinátákat használva számoljuk ki: $x = r\cos\varphi, y = r\sin\varphi, z$, ahol $0 \le r \le \frac{1}{2}$, $0 \le \varphi \le 2\pi$, $-\sqrt{1-r^2} \le z \le \sqrt{1-r^2}$, (mivel az egység sugarú gömbnél $x^2+y^2+z^2 = 1$, ezért $z = \pm\sqrt{1-(x^2+y2)} = \pm\sqrt{1-r^2}$, innen adódik, hogy $-\sqrt{1-r^2} \le z \le \sqrt{1-r^2}$). Mivel térfogatot számolunk, ezért $f(x,y,z) = 1$, így

$$V_{\text{átfúrt}} = \int_{r=0}^{1/2}\left(\int_{\varphi=0}^{2\pi}\left(\int_{z=-\sqrt{1-r^2}}^{\sqrt{1-r^2}}1\cdot rdz\right)d\varphi\right)dr = \int_{r=0}^{1/2}\left(\int_{\varphi=0}^{2\pi}2r\sqrt{1-r^2}d\varphi\right)dr =$$

$$\int_{r=0}^{1/2} 4\pi r \sqrt{1-r^2}dr = \left[-2\pi\frac{(1-r^2)^{3/2}}{3/2}\right]_0^{1/2} = \frac{4\pi}{3} - \frac{4\pi}{3}\left(\frac{3}{4}\right)^{3/2},$$

így

$$V_{\text{megmaradó}} = V_{\text{gömb}} - V_{\text{átfúrt}} = \frac{4\pi}{3} - \left(\frac{4\pi}{3} - \frac{4\pi}{3}\left(\frac{3}{4}\right)^{3/2}\right) = \frac{4\pi}{3}\left(\frac{3}{4}\right)^{3/2}$$