Komplex számok

A valós számok körében az $x^2 = -1$ egyenletnek nincs megoldása. Jelölje „i" azt a számot, amelyre az teljesül, hogy $i^2 = -1$. Ez az i egy új, nem valós szám.

Legyen $a, b \in \mathbb{R}$

A $z = a + bi$ alakú számokat komplex számoknak hívjuk. Komplex számok halmaza: $\mathbb{C}$.

Pl. $z = 5 + 2i$

$z = a + bi$, a: valós rész
b: képzetes rész

A $z = a + bi$ alakban megadott komplex számot a komplex szám algebrai alakjának hívjuk.

$z = 4 - 3i$: $a = 4$, $b = -3$

A $z = a + bi$ komplex szám konjugáltja: $\bar{z} = a - bi$

Pl. $z = 9 + 2i$: $\bar{z} = 9 - 2i$
$z = 4 - 5i$: $\bar{z} = 4 + 5i$

Műveletek komplex számokkal

$z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$

$z_1 + z_2 = (a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2) i$

$z_1 - z_2 = (a_1 + b_1 i) - (a_2 + b_2 i) = (a_1 - a_2) + (b_1 - b_2) i$

$$z_1 z_2 = (a_1 + b_1 i)(a_2 + b_2 i) = a_1 a_2 + a_1 b_2 i + a_2 b_1 i + b_1 b_2 i^2 =$$
$$a_1 a_2 - b_1 b_2 + (a_1 b_2 + a_2 b_1) i$$

Pl. $z_1 = 4+3i$

$z_2 = 8-5i$

$$z_1 + z_2 = 4+3i + 8-5i = 12-2i$$
$$z_1 - z_2 = (4+3i) - (8-5i) = -4 + 8i$$
$$z_1 z_2 = (4+3i)(8-5i) = 32 - 20i + 24i - 15\overset{-1}{i^2} =$$
$$32 - 20i + 24i + 15 = 47 + 4i$$

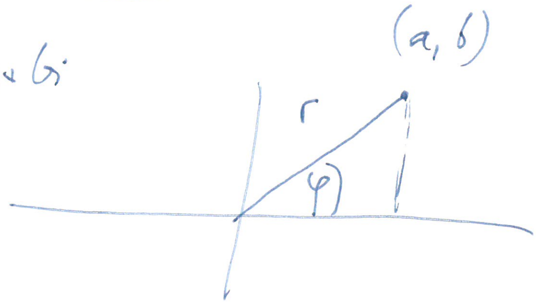
$\dfrac{z_1}{z_2}$ = algebrai alakban?

$$\frac{z_1}{z_2} = \frac{a_1 + b_1 i}{a_2 + b_2 i} = \frac{z_1 \bar{z}_2}{z_2 \bar{z}_2} = \frac{(a_1 + b_1 i)(a_2 - b_2 i)}{(a_2 + b_2 i)(a_2 - b_2 i)} = \frac{a_1 a_2 - a_1 b_2 i + a_2 b_1 i - b_1 b_2 \overset{-1}{i^2}}{a_2^2 - a_2 b_2 i + a_2 b_2 i - b_2^2 \underset{-1}{i^2}}$$
$$= \frac{a_1 a_2 + b_1 b_2 + (a_2 b_1 - a_1 b_2) i}{a_2^2 + b_2^2} = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + \frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2} i$$

Pl. $$\frac{4+3i}{8-5i} = \frac{(4+3i)(8+5i)}{(8-5i)(8+5i)} = \frac{32 + 20i + 24i + 15\overset{-1}{i^2}}{64 + 40i - 40i - 25\underset{-1}{i^2}} = \frac{17+44i}{89} =$$
$$\frac{17}{89} + \frac{44}{89} i$$

Trigonometrikus alak

$z = a + bi$



$a = r \cos \varphi$

$b = r \sin \varphi$

$$z = a + bi = r\cos\varphi + r\sin\varphi\, i = r(\cos\varphi + i \sin\varphi).$$

$0 \le \varphi \le 2\pi$

komplex szám trigonometrikus alakja

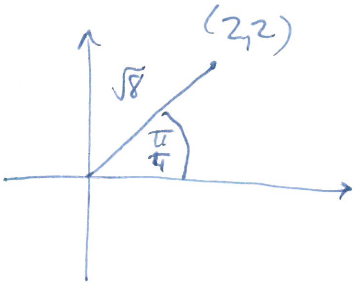
$r=\sqrt{a^2+b^2}$

$\operatorname{tg}\varphi=\frac{b}{a}$.

Pl. Írjuk trigonometrikus alakba a $z=a+bi$ algebrai alakban adott komplex számot!

a) $z=2+2i$ $\quad a=2, b=2 \quad r=\sqrt{4+4}=\sqrt{8}$

$\operatorname{tg}\varphi=\frac{2}{2}=1 \Rightarrow \underline{\varphi=\frac{\pi}{4}}$ v. $\frac{\pi}{4}+\pi=\frac{5\pi}{4}$

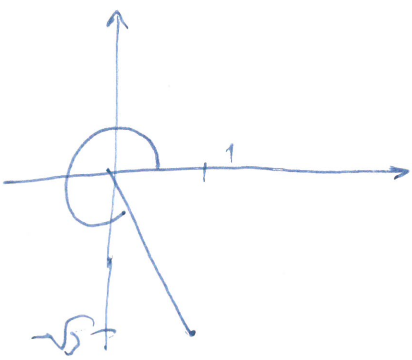


$z=2+2i=\sqrt{8}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

b) $z=1-\sqrt{3}i$: $a=1, b=-\sqrt{3}$

$r=\sqrt{1+3}=2$

$\operatorname{tg}\varphi=\frac{-\sqrt{3}}{1}=-\sqrt{3} \Rightarrow \varphi=\frac{2\pi}{3}$ v. $\frac{2\pi}{3}+\pi=\underline{\underline{\frac{5\pi}{3}}}$

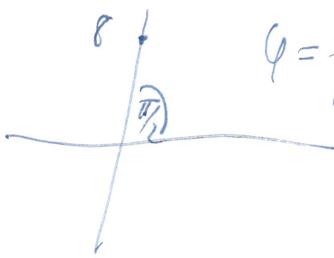


$z=2\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)$

c) $z=8i$: $a=0, b=8$

$r=\sqrt{0^2+8^2}=8$

$\operatorname{tg}\varphi=\frac{8}{0}$ nem értelmes



$\varphi=\frac{\pi}{2}$

$z=8\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

Műveletek trigonometrikus alakokkal

$$z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$$

$$z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$$

$$z_1 z_2 = r_1(\cos\varphi_1 + i\sin\varphi_1)\, r_2(\cos\varphi_2 + i\sin\varphi_2) =$$

$$r_1 r_2(\cos\varphi_1\cos\varphi_2 + i\sin\varphi_1\cos\varphi_2 + i\cos\varphi_1\sin\varphi_2 + \overset{-1}{i^2}\sin\varphi_1\sin\varphi_2)$$

$$= r_1 r_2(\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2 + i(\sin\varphi_1\cos\varphi_2 + \sin\varphi_2\cos\varphi_1))$$

Tudjuk: $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$

$$= r_1 r_2(\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2))$$

Így $z = r(\cos\varphi + i\sin\varphi)$ esetén

$$z^2 = r^2(\cos 2\varphi + i\sin 2\varphi)$$

$$z^3 = z^2 \cdot z = r^2(\cos 2\varphi + i\sin 2\varphi)\, r(\cos\varphi + i\sin\varphi) =$$

$$r^3(\cos 3\varphi + i\sin 3\varphi)$$

$\vdots$

$z^n = r^n(\cos n\varphi + i\sin n\varphi)$ ha $n \in \mathbb{Z}^+$.

$\boxed{2.\text{ea}}$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\varphi_1 + i\sin\varphi_1)}{r_2(\cos\varphi_2 + i\sin\varphi_2)} =$$

$$\frac{r_1(\cos\varphi_1 + i\sin\varphi_1)\, r_2(\cos\varphi_2 - i\sin\varphi_2)}{r_2(\cos\varphi_2 + i\sin\varphi_2)\, r_2(\cos\varphi_2 - i\sin\varphi_2)} =$$

$$\frac{r_1}{r_2}\,\frac{\cos\varphi_1\cos\varphi_2+i\sin\varphi_1\cos\varphi_2-i\sin\varphi_2\cos\varphi_1-\overbrace{i^2}^{-1}\sin\varphi_1\sin\varphi_2}{\cos^2\varphi_2-i\cos\varphi_2\sin\varphi_2+i\cos\varphi_2\sin\varphi_2-\underbrace{i^2}_{-1}\sin^2\varphi_2}=$$

$$\frac{r_1}{r_2}\,\frac{\cos\varphi_1\cos\varphi_2+\sin\varphi_1\sin\varphi_2+i(\sin\varphi_1\cos\varphi_2-\sin\varphi_2\cos\varphi_1)}{\cos^2\varphi_2+\sin^2\varphi_2}=$$

$$=\frac{r_1}{r_2}$$

Tudjuk: $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\sin\beta\cos\alpha$

$$=\frac{r_1}{r_2}\left(\cos(\alpha-\beta)+i\sin(\alpha-\beta)\right)$$

Innen $\dfrac{1}{z}=\dfrac{1}{r(\cos\varphi+i\sin\varphi)}=\dfrac{1(\cos 0+i\sin 0)}{r(\cos\varphi+i\sin\varphi)}=$

$\dfrac{1}{r}\left(\cos(-\varphi)+i\sin(-\varphi)\right)=r^{-1}\left(\cos(-\varphi)+i\sin(-\varphi)\right)$

ezért $z=r(\cos\varphi+i\sin\varphi)$ esetén

$z^{-n}=r^{-n}\left(\cos(-n\varphi)+i\sin(-n\varphi)\right).$

Összefoglalva $z=r(\cos\varphi+i\sin\varphi)$ esetén

$z^n=r^n(\cos n\varphi+i\sin n\varphi)$ ha $n\in\mathbb{Z}$

Gyökvonás: A $z=r(\cos\varphi+i\sin\varphi)$ komplex számnak a $w=\rho(\cos\alpha+i\sin\alpha)$ komplex szám n-edik gyöke, ha $z=w^n$. Így

$$r(\cos\varphi+i\sin\varphi)=z=w^n=\rho^n(\cos n\alpha+i\sin n\alpha),$$

ahonnan

$$\rho^n=r \Rightarrow \rho=r^{1/n} \quad \text{és}$$

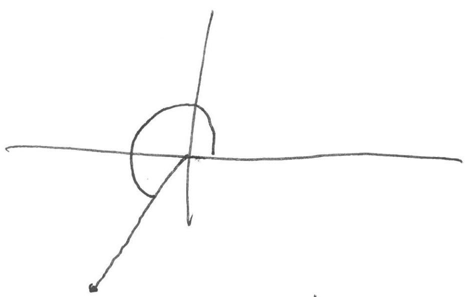
$$n\alpha=\varphi+2k\pi \Rightarrow \alpha=\frac{\varphi+2k\pi}{n} \qquad k\in\mathbb{Z}$$

Csak $k=0,1,2,\dots,n-1$ esetén kapunk különböző komplex számokat, így a $z\neq 0$ komplex számnak n db n-edik gyöke van:

$$z^{1/n}=r^{1/n}\left(\cos\frac{\varphi+2k\pi}{n}+i\sin\frac{\varphi+2k\pi}{n}\right) \qquad k=0,1,2,\dots,n-1$$

Pl. 1. $z=-128-128\sqrt{3}i$, $z^{1/4}=?$ algebrai alakban!

$$a=-128,\ b=-128\sqrt{3} \Rightarrow r=\sqrt{(-128)^2+(-128\sqrt{3})^2}=256$$



$$\varphi=\frac{-128\sqrt{3}}{-128}=\sqrt{3}\Rightarrow \varphi=\frac{\pi}{3} \text{ v. } \underline{\underline{\frac{4\pi}{3}}}$$

$$z^{1/4}=256^{1/4}\left(\cos\frac{\frac{4\pi}{3}+2k\pi}{4}+i\sin\frac{\frac{4\pi}{3}+2k\pi}{4}\right)$$

$k=0,1,2,3$

$k=0$: $z_1=256^{1/4}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=4\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=2+2\sqrt{3}i$

$k=1$: $z_2=256^{1/4}\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)=4\left(-\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)=-2\sqrt{3}+2i$

$k=2$: $z_3=256^{1/4}\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)=4\left(-\frac{1}{2}+i\left(-\frac{\sqrt{3}}{2}\right)\right)=-2-2\sqrt{3}i$

$k=3$: $z_4=256^{1/4}\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)=4\left(\frac{\sqrt{3}}{2}+i\left(-\frac{1}{2}\right)\right)=2\sqrt{3}-2i$

2, $z^2+4z+13=0$

$$z_{1,2}=\frac{-4\pm\sqrt{16-52}}{2}=\frac{-4\pm\sqrt{-36}}{2}$$

$\sqrt{-36}$: $\quad -36=36(\cos\pi+i\sin\pi)$

$$\sqrt{-36}=36^{1/2}\left(\cos\frac{\pi+2k\pi}{2}+i\sin\frac{\pi+2k\pi}{2}\right) \qquad k=0,1$$

$k=0$: $36^{1/2}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=6i$

$k=1$: $36^{1/2}\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)=-6i$

$$z_{1,2}=\frac{-4\pm(\pm 6i)}{2}=\frac{-4\pm 6i}{2}\begin{cases}-2+3i\\-2-3i\end{cases}$$