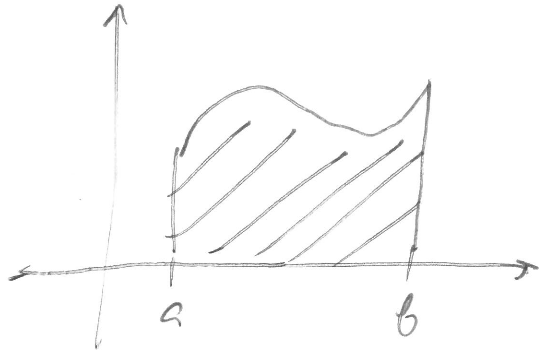
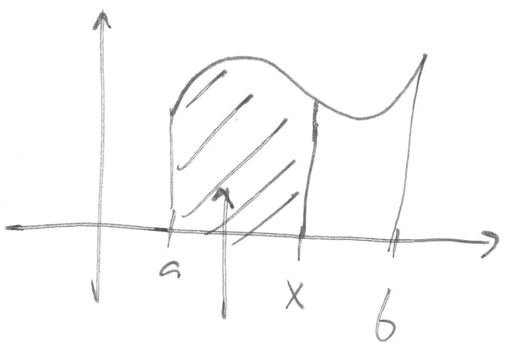
INTEGRÁLSZÁMÍTÁS

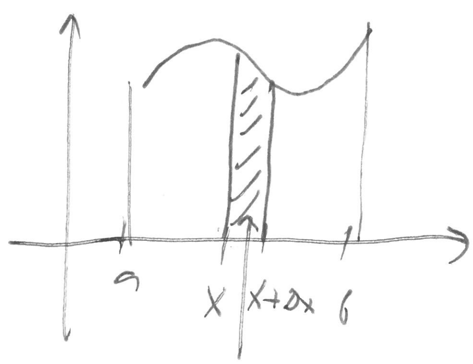
Feladat: Legyen $f(x)$ egy folytonos és pozitív fv $[a,b]$-n. Határozzuk meg az $f(x)$ grafikonja alatti területet!



ter = ?



$ter = ter(x)$



$$ter = ter(x+\Delta x) - ter(x)$$
$$\approx f(x)\Delta x$$

$$\Rightarrow f(x) \approx \frac{ter(x+\Delta x) - ter(x)}{\Delta x} \approx \lim_{\Delta x \to 0} \frac{ter(x+\Delta x) - ter(x)}{\Delta x} =$$

$$= ter'(x)$$

Tehát $ter'(x) = f(x)$.

Def Az $f(x)$ fv primitív fve a $F(x)$, ha $F'(x) = f(x)$ $\forall\, x \in [a,b]$.

Megj 1. Ha $F(x)$ primitív fv $f(x)$-nek és $c \in \mathbb{R}$, akkor $F(x)+c$ is primitív fv, mert $(F(x)+c)' = F'(x) = f(x)$.

2. Ha $F(x)$ és $G(x)$ is primitív fv-e $f(x)$-nek, akkor

$$(G(x)-F(x))' = f(x) - f(x) = 0 \Rightarrow \exists c \in \mathbb{R}, \text{ hogy } G(x)-F(x)=c,$$

azaz $G(x) = F(x)+c$

Tehát ha $F(x)$ egy primitív fv-e $f(x)$-nek $[a,b]$-n, akkor az összes primitív fv-t megkapjuk $F(x)+c$, $c \in \mathbb{R}$ alakban.

Def Az $f(x)$ fv határozatlan integrálja a primitív fv-ek halmaza. Jel: $\int f(x)\,dx$

Ha $F(x)$ egy primitív fv $\Rightarrow \int f(x)\,dx = \{F(x)+c\} = F(x)+c$
$c \in \mathbb{R}$

Pl. $\int x\,dx = \frac{x^2}{2}+c$.

Visszatérünk a feladathoz: legyen $F(x)$ egy primitív fv-e $f(x)$-nek. Ekkor alkalmas c-vel $ter(x) = F(x)+c$

$c = ?$

$0 = ter(a) = F(a)+c \Rightarrow c = -F(a) \Rightarrow ter(x) = F(x)-F(a)$

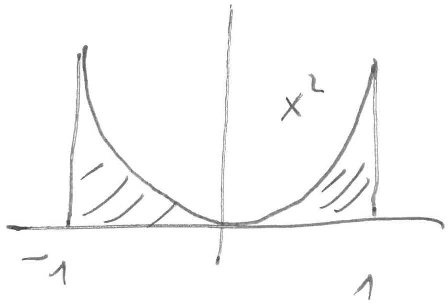
$ter = ter(b) = F(b) - F(a)$

Newton-Leibniz tétel: Legyen $f(x)$ egy folytonos, pozitív fv és $f(x)$ primitív fve $F(x)$. Ekkor

$ter = F(b) - F(a)$

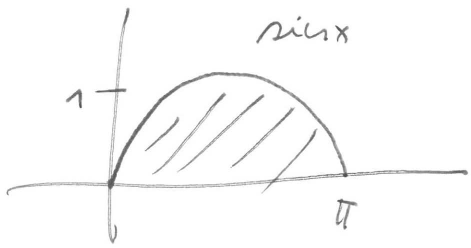
Jelölés: $\int_a^b f(x)\,dx = F(b) - F(a) = [F(x)]_a^b$

Pl. 1.



$ter = \int_{-1}^{1} x^2\,dx = \left[\frac{x^3}{3}\right] = \frac{1^3}{3} - \frac{(-1)^3}{3} = \frac{2}{3}$

2.



$ter = \int_0^{\pi} \sin x\,dx = [-\cos x]_0^{\pi} =$

$= -\cos\pi - (-\cos 0) = -(-1) + 1 = 2$

Alapintegrálok

1) $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$ ha $n \neq -1$, mert $(x^{n+1})' = (n+1)x^n$.

2) $\int \frac{1}{x}\,dx = \ln|x| + c$, mert ha $\begin{cases} x > 0 & (\ln|x|)' = (\ln x)' = \frac{1}{x} \\ x < 0 & (\ln|x|)' = (\ln(-x))' = \frac{1}{-x}(-1) = \frac{1}{x} \end{cases}$

4.) $\int \sin x\,dx = -\cos x + c$

5.) $\int \cos x\,dx = \sin x + c$

6.) $\int \frac{1}{\cos^2 x}\,dx = \operatorname{tg} x + c$

7.) $\int \frac{1}{\sin^2 x}\,dx = -\operatorname{ctg} x + c$

8.) $\int \operatorname{sh} x\,dx = \operatorname{ch} x + c$

9.) $\int \operatorname{ch} x\,dx = \operatorname{sh} x + c$

10.) $\int \frac{1}{\operatorname{ch}^2 x}\,dx = \operatorname{th} x + c$

11.) $\int \frac{1}{\operatorname{sh}^2 x}\,dx = \operatorname{cth} x + c$

12.) $\int \frac{1}{1+x^2}\,dx = \operatorname{arctg} x + c$

13.) $\int \frac{1}{1-x^2}\,dx = \begin{cases} \operatorname{arth} x & \text{ha } |x|<1 \\ \operatorname{arcth} x & \text{ha } |x|>1 \end{cases}$

14.) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + c$

15.) $\int \frac{1}{\sqrt{1+x^2}}\,dx = \operatorname{arsh} x + c$

16.) $\int \frac{1}{\sqrt{x^2-1}}\,dx = \operatorname{arch} x + c$

Integrálási szabályok

1.) Legyen $\int f(x)\,dx = F(x) + c$, $\int g(x)\,dx = G(x) + c$

$\Rightarrow \int f(x) \pm g(x)\,dx = F(x) \pm G(x) + c$, mert

$(F(x) \pm G(x))' = F'(x) \pm G'(x) = f(x) \pm g(x)$

Pl. $\int x^3 + \sin x\,dx = \frac{x^4}{4} + (-\cos x) + c$

2.) Legyen $\int f(x)\,dx = F(x) + c$ és $A \in \mathbb{R} \Rightarrow \int A f(x)\,dx = A F(x) + c$,

mert $(A F(x))' = A F'(x) = A f(x)$.

Pl. $\int 2x^5\,dx = 2 \cdot \frac{x^6}{6} + c$

3.) Legyen $\int f(x)dx = F(x)+c$. Ekkor $\int f(ax+b)dx = \frac{F(ax+b)}{a}+c$, mert $\left(\frac{F(ax+b)}{a}\right)' = \frac{1}{a}F'(ax+b)\cdot a = F'(ax+b) = f(ax+b)$.

Pl. $\int (3x-2)^7 dx = \frac{\frac{(3x-2)^8}{8}}{3}+c = \frac{(3x-2)^8}{24}+c$

$\int x^7 dx = \frac{x^8}{8}+c$

4.) $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)|+c$, mert

$$(\ln|f(x)|)' = \begin{cases} (\ln f(x))' = \frac{1}{f(x)}f'(x) = \frac{f'(x)}{f(x)} & \text{ha } f(x)>0 \\ (\ln(-f(x)))' = \frac{1}{-f(x)}(-f'(x)) = \frac{f'(x)}{f(x)} & \text{ha } f(x)<0. \end{cases}$$

Pl. $\int \operatorname{ctg} x\, dx = \int \frac{\cos x}{\sin x}dx = \int \frac{(\sin x)'}{\sin x}dx = \ln|\sin x|+c$

5.) $\int f^{\mu}(x) f'(x)\, dx = \frac{f^{\mu+1}(x)}{\mu+1}+c$ ha $\mu \neq -1$, mert

$\left(\frac{f^{\mu+1}(x)}{\mu+1}\right)' = \frac{1}{\mu+1}(\mu+1)\cdot f^{\mu}(x) f'(x) = f^{\mu}(x) f'(x)$.

Pl. $\int 2x\sqrt{x^2+1}\, dx = \int (x^2+1)^{1/2}(x^2+1)' dx = \frac{(x^2+1)^{3/2}}{3/2}+c$.

6.) Parciális integrálás: $(uv)' = u'v+uv' \Rightarrow$

$\int (u(x)v(x))' dx = \int u'(x)v(x)dx + \int u(x)v'(x)\,dx \Rightarrow$

$\int u(x)v'(x)dx = \int (u(x)v(x))' dx - \int u'(x)v(x)dx$

$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$

$\int u\cdot v' = u\cdot v - \int u'v$

Re. 1. $\int x e^{2x}\,dx = x\cdot\frac{e^{2x}}{2} - \int 1\cdot\frac{e^{2x}}{2}\,dx = x\cdot\frac{e^{2x}}{2} - \frac{e^{2x}}{4} + c$ (6)

$u\ v'$

$u'=1\quad v=\frac{e^{2x}}{2}$

2. $\int (x^2+2x-3)\sin 3x\,dx = -(x^2+2x-3)\cdot\frac{\cos 3x}{3} - \int (2x+2)\left(-\frac{\cos 3x}{3}\right)dx =$

$u\quad v'\qquad\qquad u\quad v'$

$u'=2x+2\quad v=-\frac{\cos 3x}{3}\qquad u'=2\quad v=-\frac{\sin 3x}{9}$

$-(x^2+2x-3)\cdot\frac{\cos 3x}{3} - \left((2x+2)\left(-\frac{\sin 3x}{9}\right) - \int 2\left(-\frac{\sin 3x}{9}\right)dx\right) =$

$-(x^2+2x-3)\frac{\cos 3x}{3} + (2x+2)\frac{\sin 3x}{9} + \frac{2}{27}\cos 3x + c$

3. $\int x^2 \operatorname{arctg} x\,dx = \frac{x^3}{3}\operatorname{arctg} x - \int \underbrace{\frac{x^3}{3}\cdot\frac{1}{1+x^2}}\,dx =$

$v'\quad u$

$v=\frac{x^3}{3}\quad u'=\frac{1}{1+x^2}\qquad \frac{1}{3}\frac{x^3}{1+x^2} = \frac{1}{3}\frac{x^3+x-x}{1+x^2} = \frac{1}{3}\left(x-\frac{x}{1+x^2}\right)$

$\frac{x^3}{3}\operatorname{arctg} x - \int \frac{1}{3}x - \frac{1}{3}\frac{1}{2}\frac{2x}{1+x^2}\,dx = \frac{x^3}{3}\operatorname{arctg} x - \frac{1}{3}\frac{x^2}{2} + \frac{1}{6}\cdot\ln(1+x^2) + c$

4. $\int \ln x\,dx = \int 1\cdot\ln x\,dx = x\ln x - \int \underbrace{x\cdot\frac{1}{x}}_{1}\,dx = x\ln x - x + C$

$v'\ u$

$v=x\quad u'=\frac{1}{x}$

5. $\int e^{-x}\cos 3x\,dx = -e^{-x}\cos 3x - \int (-e^{-x})(-3\sin 3x)\,dx =$

$v'\quad u\qquad\qquad v'\quad u$

$u'=-3\sin 3x\qquad v=e^{-x}\quad u'=-9\cos 3x$

$v=-e^{-x}$

$-e^{-x}\cos 3x - \left(-3\sin 3x\cdot e^{-x} - \int -9e^{-x}\cos 3x\right) =$

$-e^{-x}\cos 3x + 3\sin 3x\cdot e^{-x} - 9\int e^{-x}\cos 3x\,dx$

$10\int e^{-x}\cos 3x\,dx = -e^{-x}\cos 3x + 3\sin 3x\, e^{-x}$

$\int e^{-x}\cos 3x\,dx = -\frac{1}{10}e^{-x}\cos 3x + \frac{3}{10}e^{-x}\cdot\sin 3x$

Általános szabály:

$$p_n(x)=a_nx^n+a_{n-1}x^{n-1}+\dots+a_1x+a_0$$

I. $$\int \underbrace{p_n(x)}_{u}\underbrace{\begin{Bmatrix} e^{ax+b}\\ \sin(ax+b)\\ \cos(ax+b)\end{Bmatrix}}_{v'}dx$$

II. $$\int \underbrace{p_n(x)}_{v'}\underbrace{\begin{Bmatrix} \ln(ax+b)\\ \arcsin(ax+b)\\ \arccos(ax+b)\\ \operatorname{arctg}(ax+b)\\ \operatorname{arcctg}(ax+b)\\ \operatorname{arsh}(ax+b)\\ \operatorname{arch}(ax+b)\\ \operatorname{arth}(ax+b)\\ \operatorname{arcth}(ax+b)\end{Bmatrix}}_{u}dx$$

III. $$\int e^{ax+b}\begin{Bmatrix}\sin(\alpha x+\beta)\\ \cos(\alpha x+\beta)\end{Bmatrix}dx$$

Kétszer integrálunk majd kifejezzük az integrált

7. Racionális törtfüggvények integrálása

$$p_n(x)=a_nx^n+a_{n-1}x^{n-1}+\dots+a_1x+a_0$$

$$q_m(x)=b_mx^m+b_{m-1}x^{m-1}+\dots+b_1x+b_0$$

$$\int \frac{p_n(x)}{q_m(x)}\,dx=\dots$$

Ha $n\geq m$, akkor elosztjuk $p_n(x)$-et $q_m(x)$-mel.

Polinomok osztása:

Pl. 1. $x^4+x^3-3x+2 : x+4 = x^3-3x^2+12x-51$

$$-\quad x^4+4x^3$$
$$\overline{\qquad -3x^3}$$
$$-\quad -3x^3-12x^2$$
$$\overline{\qquad 12x^2-3x}$$
$$-\quad 12x^2+48x$$
$$\overline{\qquad -51x+2}$$
$$-\quad -51x-204$$
$$\overline{\qquad 206}$$

$$x^4+x^3-3x+2=(x+4)(x^3-3x^2+12x-51)+206$$

2. $x^4+4x^3+6x^2-3 : x^2+x-2 = x^2+3x+5$

$$-\quad x^4+x^3-2x^2$$
$$\overline{\qquad 3x^3+8x^2}$$
$$-\quad 3x^3+3x^2-6x$$
$$\overline{\qquad 5x^2+6x-3}$$
$$-\quad 5x^2+5x-10$$
$$\overline{\qquad x+7}$$

$$x^4+4x^3+6x^2-3=\underbrace{(x^2+3x+5)}_{\text{hányados}}(x^2+x-2)+\underbrace{x+7}_{\text{maradék}}$$

Polinomok

Algebra alaptétele: Ha $p(z)=a_nz^n+a_{n-1}z^{n-1}+\dots+a_1z+a_0$ egy komplex együtthatós polinom ($a_i\in\mathbb{C}$), akkor ennek a polinomnak van komplex gyöke, azaz létezik $z_0\in\mathbb{C}$, hogy $p(z_0)=0$.

Megj: $p(z)$ polinomot elosztjuk $z-z_0$-lal:

$$p(z)=(z-z_0)q(z)+A$$
$$0=p(z_0)=(z_0-z_0)q(z_0)+A=A \Rightarrow p(z)=(z-z_0)q(z).$$

Meg: Ha $z_0 \in \mathbb{C}$ gyöke a $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, $a_i \in \mathbb{R}$ akkor $\overline{z_0}$ is gyöke (nem biz)

valós együtthatós polinom

Így $p(z) = q(z)(z - z_0) = r(z)(z - \overline{z_0})(z - z_0) =$

$z_0 = x_0 + iy_0$, $\overline{z_0} = x_0 - iy_0$

$(z - z_0)(z - \overline{z_0}) = (z - (x_0 + iy_0))(z - (x_0 - iy_0)) =$

$z^2 - z(x_0 + iy_0 + x_0 - iy_0) + (x_0 + iy_0)(x_0 - iy_0) =$

$z^2 - 2x_0 z + x_0^2 + y_0^2$

$p(z) = r(z)(z^2 - 2x_0 z + x_0^2 + y_0^2)$

valós együtthatós

nincs valós gyöke
$p_1^2 - 4q_1 < 0$

Köv $p(x) = a_n (x - \alpha_1)^{k_1} (x - \alpha_2)^{k_2} \cdots (x - \alpha_r)^{k_r} (x^2 + p_1 x + q_1)^{l_1} \cdots$
$(x^2 + p_s x + q_s)^{l_s}$

$\alpha_1, \dots, \alpha_r$ különböző valós számok

p_i, q_i valósak, $p_i^2 - 4q_i < 0$

α_i multiplicitása: k_i

Pl. 1. $p(x) = x^3 + x^2 - x - 1 = x^2(x+1) - (x+1) = (x^2 - 1)(x+1) = (x+1)^2(x-1)$

gyökök: $\alpha_1 = -1$ kétszeres multiplicitás

$\alpha_2 = 1$ egyszeres multiplicitás

2. $p(x) = x^4 + x^2 + 1 = x^4 + 2x^2 + 1 - x^2 = (x^2+1)^2 - x^2 = (x^2 - x + 1)(x^2 + x + 1)$

3. $p(x) = x^3 + x^2 + 4x + 4 = (x^2 + 4)(x + 1)$

4. $p(x) = x^5 + 2x^3 + x = x(x^4 + 2x^2 + 1) = x(x^2 + 1)^2$

Racionális törtfüggvények integrálása

$$\int \frac{p_n(x)}{q_m(x)}\,dx$$

1. lépés: Ha $n \geq m$, akkor $p_n(x)$-et elosztjuk $q_m(x)$-mel.

$$p_n(x) = r(x)\,q_m(x) + s_k(x)$$

$$\int \frac{p_n(x)}{q_m(x)}\,dx = \int \frac{r(x)\,q_m(x) + s_k(x)}{q_m(x)}\,dx = \int r(x) + \frac{s_k(x)}{q_m(x)}\,dx,$$

ahol $s_k(x)$ egy k-adfokú polinom, $k < m$.

2. lépés: $q_m(x)$-et elsőfokú és másodfokú tényezők hatványainak szorzatára bontjuk. $q_m(x) = a(x-\alpha_1)^{k_1}\cdots(x-\alpha_r)^{k_r}(x^2+p_1x+q_1)^{l_1}\cdots(x^2+p_sx+q_s)^{l_s}$

3. lépés: $\frac{s_k(x)}{q_m(x)}$-et $\frac{A}{(x-\alpha_i)^j}$ $j=1,2,\dots,l_i$ és

$$\frac{Bx+C}{(x^2+p_ix+q_i)^h} \qquad h=1,2,\dots,l_s$$

alakú törtek összegére bontjuk

4. lépés: ezeket a résztörteket integráljuk.

Pl. 1. $\int \frac{3x+2}{x^2+x-6}\,dx$

$$x^2+x-6 = (x-2)(x+3)$$

$$x_{1,2} = \frac{-1 \pm \sqrt{1+24}}{2} = \frac{-1 \pm 5}{2} \begin{cases} 2 \\ -3 \end{cases}$$

$$\frac{3x+2}{x^2+x-6} = \frac{A}{x-2} + \frac{B}{x+3} = \frac{A(x+3)+B(x-2)}{(x-2)(x+3)} = \frac{(A+B)x+3A-2B}{(x-2)(x+3)}$$

$$3x+2=(A+B)x+3A-2B \Rightarrow A+B=3$$
$$3A-2B=2$$
$$3A+3B=9$$
$$-5B=-7 \Rightarrow B=1{,}4 \Rightarrow A=1{,}6$$

$$\int \frac{3x+2}{x^2+x-6}\,dx=\int \frac{1{,}6}{x-2}+\frac{1{,}4}{x+3}\,dx=1{,}6\ln|x-2|+1{,}4\ln|x+3|+C$$

2. $$\int \frac{4x^4+3x^3-x^2+x+5}{x^3+x^2-x-1}\,dx$$

1. $4x^4+3x^3-x^2+x+5 : x^3+x^2-x-1 = 4x-1$

$$\begin{array}{l} -\ 4x^4+4x^3-4x^2-4x \\ \hline \quad -x^3+3x^2+5x+5 \\ -\ -x^3-x^2+x+1 \\ \hline \quad 4x^2+4x+4 \end{array}$$

$$4x^4+3x^3-x^2+x+5=(4x-1)(x^3+x^2-x-1)+4x^2+4x+4$$

$$\int \frac{4x^4+3x^3-x^2+x+5}{x^3+x^2-x-1}\,dx=\int \frac{(4x-1)(x^3+x^2-x-1)+4x^2+4x+4}{x^3+x^2-x-1}\,dx=$$

$$\int 4x-1+\frac{4x^2+4x+4}{x^3+x^2-x-1}\,dx$$

2. $x^3+x^2-x-1=x^2(x+1)-(x+1)=(x^2-1)(x+1)=(x+1)^2(x-1)$

3. $$\frac{4x^2+4x+4}{(x+1)^2(x-1)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x-1}=$$

$$\frac{A(x+1)(x-1)+B(x-1)+C(x+1)^2}{(x+1)^2(x-1)}=\frac{A(x^2-1)+B(x-1)+C(x^2+2x+1)}{(x+1)^2(x-1)}$$

$$\frac{(A+C)x^2+(B+2C)x+(-A-B+C)}{(x+1)^2(x-1)}$$

$A+C=4$

$B+2C=4$

$+\ -A-B+C=4$

$4C=12 \Rightarrow C=3 \Rightarrow A=1,\ B=-2$

$$\frac{4x^2+4x+4}{(x+1)^2(x-1)}=\frac{1}{x+1}-\frac{2}{(x+1)^2}+\frac{3}{x-1}$$

4. $$\int 4x-1+\frac{1}{x+1}-\frac{2}{(x+1)^2}+\frac{3}{x-1}\,dx=2x^2-x+\ln|x+1|+\frac{2}{x+1}$$

$$+3\ln|x-1|+C.$$

3. $$\int \frac{2x+4}{x^3-1}\,dx$$

$x^3-1=(x-1)(x^2+x+1)$

$$\frac{2x+4}{x^3-1}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}=\frac{A(x^2+x+1)+(Bx+C)(x-1)}{x^3-1}$$

$$\frac{(A+B)x^2+(A+C-B)x+A-C}{x^3-1}$$

$$\left.\begin{aligned} A+B&=0\\ A+C-B&=2\\ A-C&=4\end{aligned}\right\}\ (+)\ 3A=6 \Rightarrow A=2 \Rightarrow B=-2,\ C=-2$$

$$\int \frac{2x+4}{x^3-1}\,dx = \int \frac{2}{x-1} + \frac{-2x-2}{x^2+x+1}\,dx = \int \frac{2}{x-1} + \int \frac{-2x-2}{x^2+x+1}\,dx =$$

$$\int \frac{2}{x-1}\,dx = 2\ln|x-1| + c$$

$$\int \frac{-2x-2}{x^2+x+1} = \int \frac{-(2x+1)-1}{x^2+x+1}\,dx = \int -\frac{2x+1}{x^2+x+1} - \int \frac{1}{x^2+x+1}\,dx =$$

$$-\ln|x^2+x+1| - \frac{2}{\sqrt{3}}\,\mathrm{arctg}\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right) + c$$

$$\int \frac{1}{x^2+x+1}\,dx = \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx = \frac{1}{\frac{3}{4}} \int \frac{1}{1+\frac{(x+1/2)^2}{3/4}}\,dx =$$

$$\frac{4}{3}\int \frac{1}{1+\frac{4}{3}\left(x+\frac{1}{2}\right)^2}\,dx = \frac{4}{3}\int \frac{1}{1+\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)^2}\,dx =$$

$$\frac{4}{3}\;\frac{\mathrm{arctg}\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)}{2/\sqrt{3}} + c = \frac{2}{\sqrt{3}}\,\mathrm{arctg}\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right) + c$$

$$= 2\ln|x-1| - \ln|x^2+x+1| - \frac{2}{\sqrt{3}}\,\mathrm{arctg}\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right) + c.$$

8. $\sin x$ és $\cos x$ hatványai, szorzatai

1. $\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \int \frac{1}{2} + \frac{1}{2}\cos 2x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + c$

2. $\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \int \frac{1}{2} - \frac{1}{2}\cos 2x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

3. $\int \sin 2x \cos 3x\,dx = \int \frac{1}{2}\left(\sin 5x + \sin(-x)\right)dx = \int \frac{1}{2}\sin 5x - \frac{1}{2}\sin x\,dx$

$\sin\alpha \cdot \cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$

$= -\frac{1}{2}\,\frac{\cos 5x}{5} + \frac{1}{2}\cos x + c$

4. $\int \cos^3 x\, dx = \int \cos^2 x \cdot \cos x \int \frac{1+\cos 2x}{2} \cos x\, dx =$

$\int \frac{1}{2} \cos x + \frac{1}{2} \overset{\beta}{\widehat{\cos x}}\, \overset{\alpha}{\widehat{\cos 2x}}\, dx = \int \frac{1}{2} \cos x + \frac{1}{2} \cdot \frac{1}{2} (\cos 3x + \cos x)\, dx =$

$\cos\alpha \cos\beta = \frac{1}{2} (\cos(\alpha+\beta) + \cos(\alpha-\beta))$

$\int \frac{3}{4} \cos x + \frac{1}{4} \cos 3x\, dx = \frac{3}{4} \sin x + \frac{1}{4} \cdot \frac{\sin 3x}{3} + C$

5.) $\int \cos^4 x\, dx = \int (\cos^2 x)^2 dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx = \int \frac{1+2\cos 2x + \cos^2 2x}{4} dx$

$= \int \frac{1}{4} + \frac{1}{2} \cos 2x + \frac{1}{4} \cos^2 2x\, dx = \int \frac{1}{4} + \frac{1}{2} \cos 2x + \frac{1}{4} \cdot \frac{1+\cos 4x}{2} dx =$

$\int \frac{3}{8} + \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x\, dx = \frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C$

6.) $\int \cos x \cdot \sin^2 x\, dx = \frac{\sin^3 x}{3} + C$

7.) $\int \cos^5 x \sin^2 x\, dx = \int \cos x \cdot \cos^4 x \cdot \sin^2 x\, dx =$

$(\cos^2 x)^2 = (1-\sin^2 x)^2 = 1 - 2\sin^2 x + \sin^4 x$

$= \int \cos x (1 - 2\sin^2 x + \sin^4 x) \sin^2 x\, dx =$

$\int \cos x - 2 \cos x \sin^4 x + \cos x \cdot \sin^6 x\, dx =$

$\sin x - 2 \cdot \frac{\sin^5 x}{5} + \frac{\sin^7 x}{7} + C$

8.) $\int \overset{\beta}{\widehat{\sin x}}\, \overset{\alpha}{\widehat{\sin 2x}}\, \sin 3x\, dx = \int \frac{1}{2} (\cos x - \cos 3x) \sin 3x\, dx =$

$\sin\alpha \sin\beta = \frac{1}{2} (\cos(\alpha-\beta) - \cos(\alpha+\beta))$

$$= \int \frac{1}{2} \cos x \sin 3x - \frac{1}{2} \sin 3x \cos 3x \, dx =$$

$$\sin\alpha \cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$$

$$= \int \frac{1}{2}\frac{1}{2}\left(\sin 4x + \sin 2x\right) - \frac{1}{2}\frac{1}{2}\left(\sin 6x + \cancel{\sin 0}\right) dx =$$

$$\int \frac{1}{4}\sin 2x + \frac{1}{4}\sin 4x - \frac{1}{4}\sin 6x \, dx = -\frac{1}{8}\cos 2x - \frac{1}{16}\cos 4x + \frac{1}{24}\cos 6x + c.$$

9. <u>Helyettesítés</u>

Legyen $\int f(x)dx = F(x)+c$, azaz $\frac{d}{dx}F(x) = f(x)$.

$x = g(t)$ esetén

$$\frac{d}{dt}F(g(t)) = f(g(t))\cdot g'(t), \text{ azaz } \int f(g(t))g'(t)dt = F(g(t))+c$$

Így $\int f(\underline{x})\,\underline{\underline{dx}} = F(x)+c = F(g(t))+c = \int f(\underline{g(t)})\cdot \underline{\underline{g'(t)dt}}$

ezért ha $x = g(t)$, akkor $dx = g'(t)dt$

Pl. 1. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \int \frac{e^t}{t}2t\,dt = \int 2e^t dt = 2e^t + c = 2e^{\sqrt{x}} + c$

$\sqrt{x} = t$

$x = t^2$

$dx = 2t\,dt$

2. $\int x\sqrt[3]{1+x}\,dx = \int (t^3-1)t\,3t^2 dt = \int 3t^6 - 3t^3 dt =$

$t = \sqrt[3]{1+x}$ $\qquad \frac{3}{7}t^7 - \frac{3}{4}t^4 + c =$

$t^3 = 1+x$ $\qquad \frac{3}{7}(1+x)^{7/3} - \frac{3}{4}(1+x)^{4/3} + c.$

$x = t^3 - 1 \Rightarrow dx = 3t^2 dt$

Speciális helyettesítések

1. Ha $f(x)$ e^x-től függ, azaz $f(x)=R(e^x)$. Ekkor

$$t=e^x \Rightarrow x=\ln t \Rightarrow \frac{dx}{dt}=\frac{1}{t},\quad dx=\frac{1}{t}\,dt$$

Pl. 1. $$\int \frac{e^x}{1+e^{2x}}\,dx=\int \frac{e^x}{1+(e^x)^2}\,dx=\int \frac{t}{1+t^2}\,\frac{1}{t}\,dt=$$

$t=e^x$

$x=\ln t$

$dx=\frac{1}{t}\,dt$

$$=\int \frac{1}{1+t^2}\,dt=\operatorname{arctg} t+c=\operatorname{arctg} e^x+c$$

2. $$\int e^{2x}\ln(e^x+1)\,dt=\int t^2\ln(1+t)\,\frac{1}{t}\,dt=\int \underbrace{t}_{v'}\,\underbrace{\ln(1+t)}_{u}\,dt=$$

$v=\frac{t^2}{2}$ $\quad u'=\frac{1}{1+t}$

$e^x=t$

$x=\ln t$

$dx=\frac{1}{t}\,dt$

$$\frac{t^2}{2}\ln(1+t)-\int \frac{t^2}{2}\,\frac{1}{1+t}\,dt=\frac{t^2}{2}\ln(1+t)-\frac{1}{2}\int \frac{t^2}{1+t}\,dt=$$

$$\frac{t^2}{2}\ln(1+t)-\frac{1}{2}\int \frac{t^2-1+1}{1+t}\,dt=\frac{t^2}{2}\ln(1+t)-\frac{1}{2}\int \frac{(t-1)(t+1)+1}{1+t}\,dt$$

$$=\frac{t^2}{2}\ln(1+t)-\frac{1}{2}\int t-1+\frac{1}{1+t}\,dt=\frac{t^2}{2}\ln(1+t)-\frac{1}{2}\left(\frac{t^2}{2}-t+\ln(1+t)\right)+c$$

$$=\frac{e^{2x}}{2}\ln(1+e^x)-\frac{1}{4}e^{2x}+\frac{1}{2}e^x-\frac{1}{2}\ln(1+e^x)+C$$

2. $f(x) = R\left(\sqrt[n]{ax+b}\right)$

$t = \sqrt[n]{ax+b}$

$t^n = ax+b$

$x = \frac{t^n}{a} - \frac{b}{a}$

$dx = \frac{nt^{n-1}}{a}\,dt$

Pl. 1. $\int \frac{dx}{1+\sqrt{2x-1}} = \int \frac{t}{1+t}\,dt = \int \frac{t+1-1}{t+1}\,dt = \int 1 - \frac{1}{t+1}\,dt =$

$t = \sqrt{2x-1}$ $\quad = t - \ln|1+t| + C = \sqrt{2x-1} - \ln\left(1+\sqrt{2x-1}\right) + C$

$t^2 = 2x-1$

$2x = t^2+1$

$x = \frac{1}{2}t^2 + \frac{1}{2}$

$dx = t\,dt$

2. $\int \frac{x}{\sqrt[3]{x+1}-1}\,dx = \int \frac{t^3-1}{t-1}\,3t^2\,dt = \int t^2+t+1\,dt =$

$t = \sqrt[3]{x+1}$ $\quad \frac{t^3}{3} + \frac{t^2}{2} + t + C = \frac{x+1}{3} + \frac{(x+1)^{2/3}}{2} + (x+1)^{1/3} + C$

$t^3 = x+1$

$x = t^3 - 1$

$dx = 3t^2\,dt$

3. $f(x) = R\left(\sqrt[n]{\frac{ax+b}{cx+d}}\right)$ $\qquad t = \sqrt[n]{\frac{ax+b}{cx+d}}$

Pl. $\int \sqrt{\frac{x-3}{x+1}}\,dx = \int \frac{8t^2}{(1-t^2)^2}\,dt = \int \frac{8t^2}{(1-t)^2(1+t)^2}\,dt$

$t = \sqrt{\frac{x-3}{x+1}}$ $\qquad (x+1)t^2 = x-3$ $\qquad x = \frac{t^2+3}{1-t^2}$

$xt^2 + t^2 = x-3$

$t^2 = \frac{x-3}{x+1}$ $\qquad t^2+3 = x(1-t^2)$

$$dx = \frac{2t(1-t^2)-(t^2+3)(-2t)}{(1-t^2)^2} = \frac{8t}{(1-t^2)^2}$$

$$\frac{8t^2}{(1-t)^2(1+t^2)} = \frac{A}{1-t} + \frac{B}{(1-t)^2} + \frac{C}{1+t} + \frac{D}{(1+t)^2} =$$

$$\frac{A(1-t)(1+t)^2 + B(1+t)^2 + C(1-t)^2(1+t) + D(1-t)^2}{(1-t)^2(1+t)^2} =$$

$$\frac{A(1+t-t^2-t^3) + B(1+2t+t^2) + C(1-t-t^2+t^3) + D(1-2t+t^2)}{(1-t)^2(1+t^2)} =$$

$$\frac{t^3(-A+C) + t^2(-A+B-C+D) + t(A+2B-C-2D) + A+B+C+D}{(1-t)^2(1+t)^2}$$

$$\left.\begin{array}{ll}
t^3: & -A+C=0 \Rightarrow A=C \\
t^2: & -A+B-C+D=8 \Rightarrow 2B-2A=8 \\
t: & A+2B-C-2D=0 \Rightarrow 2B-2D=0 \Rightarrow B=D \\
\text{konst:} & A+B+C+D=0 \Rightarrow 2A+2B=0
\end{array}\right\} \overset{\oplus}{\Rightarrow} 4B=8$$

$$B=2=D$$

$$A=-2=C$$

$$\int \frac{8t^2}{(1-t)^2(1+t)^2}\,dt = \int \frac{-2}{1-t} + \frac{2}{(1-t)^2} + \frac{-2}{1+t} + \frac{2}{(1+t)^2}\,dt =$$

$$2\ln|1-t| + \frac{2}{1-t} - 2\ln|1+t| - \frac{2}{1+t} + C =$$

$$2\ln\left|1-\sqrt{\frac{x-3}{x+1}}\right| + \frac{2}{1-\sqrt{\frac{x-3}{x+1}}} - 2\ln\left|1+\sqrt{\frac{x-3}{x+1}}\right| - \frac{2}{1+\sqrt{\frac{x-3}{x+1}}} + C$$

4.) $f(x) = R\left(\sqrt{a^2-x^2}\right)$ $\qquad x = a \sin t$

$$dx = a \cos t \, dt$$

Př. 1. $\int x^2 \sqrt{1-x^2}\, dx = \int \sin^2 t \cdot \cos t \cdot \cos t \, dt = \int \frac{1-\cos 2t}{2} \cdot \frac{1+\cos 2t}{2} \, dt$

$x = \sin t$ $\qquad \sqrt{1-x^2} = \sqrt{1-\sin^2 t} = \sqrt{\cos^2 t} = \cos t$

$dx = \cos t \, dt$

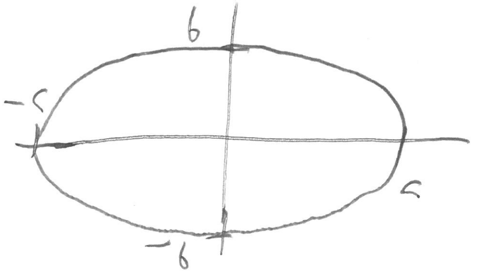
$$= \int \frac{1-\cos^2 2t}{4} \, dt = \int \frac{1 - \frac{1+\cos 4t}{2}}{4} \, dt = \int \frac{1}{8} - \frac{1}{8} \cos 4t \, dt =$$

$$\frac{1}{8} t - \frac{1}{8} \cdot \frac{\sin 4t}{4} \, dt = \frac{1}{8} t - \frac{1}{32} 2 \cdot \sin 2t \cos 2t + C =$$

$$\frac{1}{8} \arcsin x - \frac{1}{16} \cdot 2 \sin t \cos t \left(\overbrace{\cos^2 t}^{1-\sin^2 t} - \sin^2 t\right) + C =$$

$$\frac{1}{8} \arcsin x - \frac{1}{8} x \cdot \sqrt{1-x^2} \, (1 - 2x^2) + C$$

2.



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $\quad tr = ?$

$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \Rightarrow y^2 = b^2 - \frac{b^2}{a^2} x^2$

$y = \pm \sqrt{b^2 - \frac{b^2}{a^2} x^2}$

$$tr = 4 \int_0^a \sqrt{b^2 - \frac{b^2}{a^2} x^2} \, dx = 4 \int_0^a \sqrt{\frac{b^2}{a^2}(a^2 - x^2)} \, dx = \frac{4b}{a} \int_0^a \sqrt{a^2-x^2} \, dx =$$

$x = a \sin t$

$dx = a \cos t$ $\qquad \sqrt{a^2-x^2} = \sqrt{a^2 - a^2 \sin^2 t} = a\sqrt{1-\sin^2 t} = a\sqrt{\cos^2 t} = a \cos t$

$$\frac{4b}{a} \int_0^{\pi/2} a \cos t \, a \cos t \, dt = 4ab \int_0^{\pi/2} \cos^2 t \, dt = 4ab \int_0^{\pi/2} \frac{1+\cos 2t}{2} \, dt =$$

$$2ab \left[t + \frac{\sin 2t}{2} \right]_0^{\pi/2} = ab\pi$$

20.

J.) $f(x) = R(\sqrt{a^2+x^2})$

$x = a\,sh\,t \Rightarrow \sqrt{a^2+x^2} = \sqrt{a^2+a^2 sh^2 t} = \sqrt{a^2(1+sh^2 t)} =$

$dx = a\,ch\,t\,dt \qquad a\sqrt{ch^2 t} = a\,ch\,t$

$f(x) = R(\sqrt{x^2-a^2})$

$x = a\,ch\,t \qquad \sqrt{x^2-a^2} = \sqrt{a^2 ch^2 t - a^2} = a\sqrt{ch^2 t - 1} =$

$dx = a\,sh\,t\,dt \qquad a\sqrt{sh^2 t} = a\,sh\,t$, ha $sh\,t > 0$

Pl. 1. $\int x^2\sqrt{x^2-1}\,dx = \int ch^2 t \cdot sh\,t \cdot sh\,t\,dt = \int ch^2 t \cdot sh^2 t\,dt =$

$x = ch\,t$

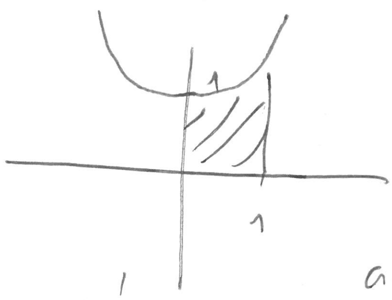
$\int \frac{1+ch\,2t}{2} \cdot \frac{ch\,2t-1}{2}\,dt = \int \frac{ch^2 2t - 1}{4}\,dt = \int \frac{\frac{1+ch\,4t}{2} - 1}{4}\,dt =$

$\int \frac{1}{8} ch\,4t - \frac{1}{8}\,dt = \frac{1}{32} sh\,4t - \frac{1}{8}t + C = \frac{1}{32} \cdot 2\,sh\,2t\,ch\,2t - \frac{1}{8}t =$

$\frac{1}{16} \cdot 2\,sh\,t\,ch\,t\,(2ch^2 t - 1) - \frac{1}{8}t + C =$

$\frac{1}{8}\sqrt{x^2-1} \cdot x\,(2x^2-1) - \frac{1}{8}\,ar\,ch\,x + C$

2.



$y^2 - x^2 = 1$, ter = ?

$y^2 = 1 + x^2$

$y = \sqrt{1+x^2}$

$ter = \int_0^1 \sqrt{1+x^2}\,dx = \int_0^{ar\,sh\,1} ch\,t \cdot ch\,t\,d = \int_0^{ar\,sh\,1} \frac{1+ch\,2t}{2}\,dt = \left[\frac{1}{2}t + \frac{1}{2}\frac{sh\,2t}{2}\right]_0^{ar\,sh\,1} =$

$x = sh\,t \qquad \sqrt{1+x^2} = ch\,t \qquad 0 \le x \le 1 \Rightarrow 0 \le sh\,t \le 1$

$dx = ch\,t$

$sh\,0 \qquad sh(ar\,sh\,1)$

$\Rightarrow 0 \le t \le ar\,sh\,1$

$$\left[\frac{1}{2}t+\frac{1}{4}\,2\,\mathrm{sh}\,t\,\underbrace{\mathrm{ch}\,t}_{\sqrt{1+\mathrm{sh}^2 t}}\right]_0^{\mathrm{arsh}\,1}=\frac{1}{2}\mathrm{arsh}\,1+\frac{1}{2}\mathrm{sh}(\mathrm{arsh}\,1)\sqrt{1+\mathrm{sh}^2(\mathrm{arsh}\,1)}-$$

$$-\left(\frac{1}{2}\cdot 0+\frac{1}{4}\cdot 2\underbrace{\mathrm{sh}\,0}_{0}\cdot\sqrt{\;\cdot\;}\right)=\frac{1}{2}\mathrm{arsh}\,1+\frac{1}{2}\cdot 1\sqrt{2}=\frac{\mathrm{arsh}\,1}{2}+\frac{\sqrt{2}}{2}$$

6. $f(x)=R(\sin x,\cos x)$

$t=\mathrm{tg}\,\frac{x}{2}$

$\frac{x}{2}=\mathrm{arctg}\,t$

$x=2\,\mathrm{arctg}\,t$

$dx=\frac{2}{1+t^2}dt$

$$\sin x=\frac{\sin 2\cdot\frac{x}{2}}{1}=\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}=\frac{2\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}+\frac{\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}}}=\frac{2\,\mathrm{tg}\,\frac{x}{2}}{1+\mathrm{tg}^2\frac{x}{2}}=\frac{2t}{1+t^2}$$

$$\cos x=\frac{\cos 2\cdot\frac{x}{2}}{1}=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}=\frac{\frac{\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}}-\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}{\frac{\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}}+\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}=$$

$$\frac{1-\mathrm{tg}^2\frac{x}{2}}{1+\mathrm{tg}^2\frac{x}{2}}=\frac{1-t^2}{1+t^2}$$

Pr. 1. $\displaystyle\int\frac{dx}{\sin x}=\int\frac{1}{\frac{2t}{1+t^2}}\,\frac{2}{1+t^2}dt=\int\frac{1}{t}dt=\ln|t|+C=$

$\ln\left|\mathrm{tg}\,\frac{x}{2}\right|+C$

2. $\displaystyle\int\frac{2}{5+4\cos x}dx=\int\frac{2}{5+4\cdot\frac{1-t^2}{1+t^2}}\,\frac{2}{1+t^2}dt=\int\frac{4}{9+t^2}dt=\int\frac{4}{9\left(1+\frac{t^2}{9}\right)}dt=$

$t=\mathrm{tg}\,\frac{x}{2}$ $\qquad\displaystyle\frac{4}{9}\int\frac{1}{1+\left(t/3\right)^2}dt=\frac{4}{9}\cdot\frac{\mathrm{arctg}\,(t/3)}{1/3}+C=\frac{4}{3}\,\mathrm{arctg}\left(\frac{1}{3}\mathrm{tg}\,\frac{x}{2}\right)+C$